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Convergence Theorems Via Hybrid Multivalued Mappings

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Abstract

It is known that algorithms are of great importance in various fields of mathematics, as they are used in finding fixed points, zeroes of metric projection points, Non-Smooth, Differential Equations, Optimization theory, and Variational Inequality problem. Accordingly, many researchers have focused on investigating and enhancing algorithms in order to utilize their potential amidst the rapid technological developments occurring in our modern world. To ensure success, effectiveness, speed, and superiority of iterative methods over other approximate methods depend on two important factors: The first is the number of iterations, and the second is time. In this paper, we introduce a new iterative method that has been generalized to a number of algorithms, which is considered a generalization of Ishikawa's iteration algorithm. We use a family of hybrid multivalued mappings, nonexpansive single-valued mappings, and $(\varphi, L)^*$- weak contraction mapping where φ is a comparison function in Hilbert space. The concept of $(\varphi, L)^*$-weak contraction mapping is a generalization of the concept (φ, L)-weak contraction mapping, and we obtain several convergence theorems under suitable conditions.

Keywords: Hybrid multivalued mapping, $(\varphi, L)^*$-weak contraction, condition (Ă) and projection operator

1. Introduction

Let Ĕ be a nonempty closed and convex subset of real Hilbert space Ȟ with norm $||.||$ and inner product $\langle .,. \rangle$. Let $\mathcal{CB}(Ĕ)$ denote the family of all nonempty closed bounded subset of Ĕ**,** while $Ǩ(Ĕ)$ denote the family of all nonempty compact subset of Ĕ . An element $ꝑ \in Ĕ$, is called fixed point ($\mathcal{FP}$) of mapping $Ť : Ĕ \longrightarrow Ĕ$ if $ꝑ = ꝑŤ$, in multivalued mapping $Ƭ : Ĕ \longrightarrow \mathcal{CB}(Ĕ)$, ꝑ is a $\mathcal{FP}$ if $ꝑ \in Ƭꝑ$. . A point ꝑ is called a common fixed point of Ť and Ƭ if $ꝑ = Ťꝑ \in Ƭꝑ$ and denoted by ($\mathcal{CFP}$). Many authors have studied extensively the $\mathcal{FP}$ theorems and the existence of $\mathcal{FP}$ of nonexpansive mappings (N-mappings), and they presented many concepts and theorems [1- 4], and other studies have examined the convergence of different iterative methods, as noted in [5-12], also they studies the equivalence of Some Iterations[13], and introduced generalization of the Mann's algorithm[14].

Defined a class of nonlinear mapping, which is called hybrid as follows:

$$||Ƭ\mathfrak{u} - Ƭ\mathfrak{z}||^2 \leq ||\mathfrak{u} - \mathfrak{z}||^2 + \langle \mathfrak{u} - Ƭ\mathfrak{u}, \mathfrak{z} - Ƭ\mathfrak{z} \rangle , \quad \forall\, \mathfrak{u}, \mathfrak{z} \in Ĕ$$

that a mapping $Ť : Ĕ \longrightarrow Ȟ$ is hybrid if:

$$3||Ƭ\mathfrak{u} - Ƭ\mathfrak{z}||^2 \leq ||\mathfrak{u} - \mathfrak{z}||^2 + ||\mathfrak{u} - Ƭ\mathfrak{z}||^2 + ||Ƭ\mathfrak{u} - \mathfrak{z}||^2 , \quad \forall\, \mathfrak{u}, \mathfrak{z} \in Ĕ$$ [15]

Stated and introduced a new concept of mapping $Ƭ : Ĕ \longrightarrow \mathcal{CB}(Ĕ)$ in Hilbert space by Hausdorff metric such that (Ƭ) is called hybrid if satisfies the following condition: $3H(Ƭ\mathfrak{u}, Ƭ\mathfrak{z})^2 \leq ||\mathfrak{u} - \mathfrak{z}||^2 + d(\mathfrak{u}, Ƭ\mathfrak{z})^2 + d(\mathfrak{z}, Ƭ\mathfrak{u})^2, \forall\, \mathfrak{u}, \mathfrak{z} \in Ĕ$, and if $\mathcal{F}(Ƭ)$ nonempty, then Ƭ is a quasi-nonexpansive[16]. The approximating $\mathcal{FP}$of (φ, L)-weak contractions ((φ, L)-W- contr) it was of attract to some researchers[17] and many scholars and researchers have made generalizations in different directions of contractive mappings;see [18-20]

The modification of Ishikawa's algorithm for two hybrid multivalued mapping in $\check{H}$:

$$\begin{cases} ȕ_1 \in \check{E} \text{ chosen arbitrarily} \\ ƶ_n = \sigma_n ȕ_n + (1-\sigma_n) Ƭ_1 ȕ_n \\ ȕ_{n+1} = \lambda_n ȕ_n + (1-\lambda_n) Ƭ_2 ƶ_n \end{cases}$$

and proved the sequence $\{ȕ_n\}$ weak converges (W- converges) to a $\mathcal{CFP}$ of $\{Ƭ_1, Ƭ_2\}$, see[21]

The hybrid algorithm was studied through[22]. After that, the focus was on convergence of the modified Picard-s hybrid iterative scheme, a Picard-S hybrid algorithm and introduced another hybrid scheme see[23-26].

In this work, we construct a new iterative scheme, that modifies the above iterative algorithm by using two hybrid multivalued mappings, two N-mappings and two $(\varphi, L)^*$-weak contraction ($(\varphi, L)^*$-W- contr) mappings in real Hilbert space.

2. Preliminaries:

We recall the following:

2.1. Definition**:** A mapping $Ƭ : \check{E} \longrightarrow \mathcal{CB}(\check{E})$ is said to be a hybrid multivalued mapping (HM-mapping) if satisfies the following condition:

$3H(Ƭȕ, Ƭƶ)^2 \leq ||ȕ - ƶ||^2 + d(ȕ, Ƭƶ)^2 + d(ƶ, Ƭȕ)^2$, $\forall\, ȕ, ƶ \in \check{E}$

and if $\mathcal{F}(Ƭ) \neq \emptyset$, then Ƭ is a quasi-nonexpansive[16].

2.2. Definition: Let $(\check{E}, d)$ be a metric space and $Ƭ: \check{E} \longrightarrow \check{E}$ any operator. Then, Ƭ is called (φ, L)-weak contraction if there exists a some $L \geq 0$ and a $\mathcal{CF}$ φ such that:

$d(Ƭȕ, Ƭƶ) \leq \varphi\, d(ȕ, ƶ) + L\, d(ƶ, Ƭȕ)$ $\forall\, ȕ, ƶ \in \check{E}$ [17]

2.3. Definition: Let $Ƭ: \check{E} \longrightarrow \check{E}$ self-mapping then Ƭ is called N- mapping if:

$||Ƭȕ - Ƭƶ|| \leq ||ȕ - ƶ||$ for all $ȕ, ƶ \in \check{E}$ [27].

2.4. Definition: A map $\varphi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is called a comparison function ($\mathcal{CF}$) if it satisfies:

(i) $r_1 < r_2 \Longrightarrow \varphi r_1 \leq \varphi r_2$ for all $r_1, r_2 \in \mathbb{R}^+$

(ii) the sequence $\{\varphi^n r\}_{n=0}^{\infty}$ converges to zero, $\forall r \in \mathbb{R}^+$such that φ^n is a stand for the n^{th} iterate of φ see[28] .

2.5. Definition: Let $(\check{E}, d)$ be a metric space and $Ƭ: \check{E} \longrightarrow \check{E}$ any operator. Then, Ƭ is called $(\varphi, L)^*$-weak contraction if there exists any a $\mathcal{CF}$ φ such that:

$d(Ƭȕ, Ƭƶ) \leq \varphi\, d(ȕ, ƶ) + L \min\{d(ƶ, Ƭȕ), d(ƶ, Ƭƶ)\}$ $\forall\, ȕ, ƶ \in \check{E}$

And if $\min\{d(ƶ, Ƭȕ), d(ƶ, Ƭƶ)\} = d(ƶ, Ƭȕ)$ then $L \geq 0$

if $\min\{d(ƶ, Ƭȕ), d(ƶ, Ƭƶ)\} = d(ƶ, Ƭƶ)$ then $0 \leq L < 1$

2.6. Remark: Clearly, the $(\varphi, L)^*$-weak contraction to be (φ, L)-weak contraction if:

$\min\{d(ƶ, Ƭȕ), d(ƶ, Ƭƶ)\} = d(ƶ, Ƭȕ)$

2.7. Definition**:** A mapping $Ƭ: \check{E} \longrightarrow \mathcal{CB}(\check{E})$ is said to satisfy Condition $(\check{A})$ if:

$||ȕ - p|| = d(ȕ, Ƭp)$ $\forall ȕ \in \check{H}$, $p \in \mathcal{F}(Ƭ)$ see[29]

2.8. Lemma**:** Let $Ƭ: \check{E} \longrightarrow \check{K}(\check{E})$ be an HM-mapping and $\{ȕ_n\}$ be a sequence in $\check{E}$ such that $ȕ_n \rightharpoonup u$ and $\lim_{n\to\infty} ||ȕ_n - ƶ_n|| = 0$ for some $ƶ_n \in Ƭȕ_n$. Then $u \in Ƭu$ [16]

2.9. Lemma**:** Let $Ƭ: \check{E} \longrightarrow \check{K}(\check{E})$ be an HM-mapping such that $\mathcal{F}(Ƭ)$ is nonempty, then $\mathcal{F}(Ƭ)$ is closed[16].

2.10. Lemma**:** Let $Ћ: Ĕ \longrightarrow Ǩ(Ĕ)$ be an HM-mapping such that $\mathcal{F}(Ћ)$ is nonempty. If Ћ satisfies Condition (Ă), then $\mathcal{F}(Ћ)$ is convex[16].

2.11. Lemma**:** For all ừ and ƶ in Ȟ and $\alpha \in [0,1]$ the following is hold:

(i) $||ừ - ƶ||^2 = ||ừ||^2 + ||ƶ||^2 - 2\langle ừ, ƶ\rangle$

(ii) $||\alpha ừ - (1-\alpha)ƶ||^2 = \alpha||ừ||^2 + (1-\alpha)||ƶ||^2 - \alpha(1-\alpha)||ừ - ƶ||^2$

(iii) If $\{ừ_n\}$ is a sequence in Ȟ such that $ừ_n \rightharpoonup ừ$, then

$\lim_{n\to\infty} \sup ||ừ_n - ƶ||^2 = \lim_{n\to\infty} \sup(||ừ_n - ừ||^2 + ||ừ - ƶ||^2)$. [30]

2.12. Lemma**:** Let $P_{Ĕ}: Ȟ \longrightarrow Ĕ$ be the metric projection from Ȟ onto Ĕ then:

$||ƶ - P_{Ĕ}\, ừ||^2 + ||ừ - P_{Ĕ}\, ừ||^2 \le ||ừ - ƶ||^2, \quad \forall\, ừ \in Ȟ$ and $ƶ \in Ĕ$ [31]

2.13. Lemma **:** Let Ĕ be a nonempty closed and convex subset of Ȟ, then the set $Ǩ = \{š \in Ĕ : ||ƶ - š||^2 \le ||ừ - š||^2 + \langle ă, š\rangle + ř\}$ is closed and convex for each $ừ, ƶ \in Ȟ$ and $ř \in \mathbb{R}$ see[32]

2.14. Lemma: Let $Ћ: Ȟ \longrightarrow Ȟ$ an N-mapping, then $\mathcal{F}(Ћ)$ is either empty or closed and convex[33]. Also from fact [34]**,** if an N-mapping $Ћ: Ȟ \longrightarrow Ȟ$ has at least one $\mathcal{FP}$, $\mathcal{F}(Ћ) \subset Ȟ$ is closed and is closed and convex and expressed as:

$$\mathcal{F}(Ћ) = \bigcap_{ừ \in Ȟ} \{ƶ \in Ȟ: \langle ừ - Ћừ, ƶ\rangle \le ||ừ||^2 - ||Ћừ||^2\}$$

3. Results and Discussion

Studied Approximating $\mathcal{FP}$ of (φ, L)-weak contractions it has attracted the interest of some researchers[18], while introduced[24] the modification for two hybrid multivalued mapping in Ȟ. Also[11] Common fixed points for hybrid pair of generalized non-expensive mappings by a three-step iterative scheme. In the other hand[34] studied Strong convergence theorems for nonexpansive mappings. In this study, a convergence theorems-W to $\mathcal{CFP}$ by multivalued maps are proved by using new algorithms.

3.1. Lemma: Let $Ћ: Ȟ \longrightarrow Ȟ$ be $(\varphi, L)^*$-W-contr- mapping where φ is a $\mathcal{CF}$ then $\mathcal{F}(Ћ)$ is nonempty.

Proof: To prove that $\mathcal{F}(Ћ) \neq \emptyset$, let $ừ_0 \in Ȟ$, a sequence $\{ừ_n\}_{n=0}^{\infty}$ defined by $ừ_{n+1} = Ћừ_n$. Since Ћ is a $(\varphi, L)^*$-W- contr, there exists a $\mathcal{CF}$ φ and some $L \ge 0$ where $\min\{d(ƶ, Ћừ), d(ƶ, Ћƶ)\} = d(ƶ, Ћừ)$, and $0 \le L < 1$ where

$\min\{d(ƶ, Ћừ), d(ƶ, Ћƶ)\} = d(ƶ, Ћƶ)$ such that:

$$d(Ћừ, Ћƶ) \le \varphi\, d(ừ, ƶ) + L \min\{d(ƶ, Ћừ), d(ƶ, Ћƶ)\}, \ \forall\, ừ, ƶ \in Ȟ \quad \textbf{(1)}$$

Take $ừ := ừ_{n-1}$, $y := ừ_n$ in (1). We obtain

$$d(ừ_n, ừ_{n+1}) \le \varphi d\left(ừ_{n-1}, ừ_n\right), \forall n = 1, 2, \ldots \quad \textbf{(2)}$$

But φ is not decreasing and from **Equation 2,** we obtain,

$d(ừ_n, ừ_{n+1}) \le \varphi^n d(ừ_0, ừ_1)$

implies that $\{ừ_n\}_{n=0}^{\infty}$ is a Cauchy sequence and by completely of Ȟ, we have $ừ_n \to ᵽ$, we shall prove that $ᵽ \in \mathcal{F}(Ћ)$. Indeed

$$\begin{aligned} d(ᵽ, Ћᵽ) &\le d\left(ᵽ, ừ_{n+1}\right) + d\left(ừ_{n+1}, Ћᵽ\right) \\ &= d(ừ_{n+1}, ᵽ) + d(Ћừ_n, Ћᵽ) \end{aligned}$$

By **Equation 1,** we obtain

$$d(Ћừ_n, Ћᵽ) \le \varphi\, d(ừ_n, ᵽ) + L \min\{d(ᵽ, Ћừ_n), d(ᵽ, Ћᵽ)\}$$

Therefore,

$$d(ᵽ, Ƭᵽ) \le d(ȕ_{n+1}, ᵽ) + \varphi\, d(ȕ_n, ᵽ) + L \min\{d(ȕ_{n+1}, ᵽ), d(ᵽ, Ƭᵽ)\} \qquad \textbf{(3)}$$

Case 1: $\min\{d(ȕ_{n+1}, ᵽ), d(ᵽ, Ƭᵽ)\} = d(ȕ_{n+1}, ᵽ)$, we have

$$d(ᵽ, Ƭᵽ) \le (1 + L) d(ȕ_{n+1}, ᵽ) + \varphi\, d(ȕ_n, ᵽ) \qquad \textbf{(4)}$$

Now suppose that $n \to \infty$ in (4), we obtain

$d(ᵽ, Ƭᵽ) = 0$ .Therefore $ᵽ \in \mathcal{F}(Ƭ)$

Case 2: $\min\{d(ȕ_{n+1}, ᵽ), d(ᵽ, Ƭᵽ)\} = d(ᵽ, Ƭᵽ)$, we have

$$(1 - L) d(ᵽ, Ƭᵽ) \le d(ȕ_{n+1}, ᵽ) + \varphi\, d(ȕ_n, ᵽ) \qquad \textbf{(5)}$$

As $n \to \infty$ in **Equation 5,** we obtain $d(ᵽ, Ƭᵽ) = 0$. Therefore, $ᵽ \in \mathcal{F}(Ƭ)$

3.2.Theorem**:** Let $Ƭ_1, Ƭ_2 : Ĕ \longrightarrow Ǩ(Ĕ)$ be HM-mapping $Ɇ_1, Ɇ_2 : Ȟ \longrightarrow Ȟ$ are an N-mapping, and $G_1, G_2 : Ȟ \longrightarrow Ȟ$ are $(\varphi, L)^*$-W- contr mapping where φ is a $\mathcal{CF}$ define by $\varphi(\mathfrak{x}) = \delta\mathfrak{x}, 0 < \delta \le 1$ such that $\Gamma \coloneqq (\bigcap_{i=1}^{2} \mathcal{F}(Ƭ_i)) \cap (\bigcap_{i=1}^{2} \mathcal{F}(Ɇ_i)) \cap (\bigcap_{i=1}^{2} \mathcal{F}(G_i)) \ne \emptyset$. Let $\{ȕ_n\}$ be a generated by:

$$\begin{cases} ȕ_1 \in Ĕ \ \text{chosen arbitrary.} \\ ƶ_n \in ŧ_n[ƀ_n G_1 ȕ_n + (1 - ƀ_n) Ƭ_1 ȕ_n] + (1 - ŧ_n) Ɇ_1 ȕ_n \\ ȕ_{n+1} \in đ_n\left[ɉ_n Ɇ_2 ȕ_n + (1 - ɉ_n) G_2 ȕ_n\right] + (1 - đ_n) Ƭ_2 ƶ_n \end{cases} \qquad \textbf{(6)}$$

For all $n \ge 1$ where $\{ŧ_n\}, \{ƀ_n\}, \{đ_n\}$ and $\{ɉ_n\} \subset (0,1]$. Assume that:

(i) $\sum_{n=0}^{\infty} \| G_1 ȕ_n - ȕ_n \| < \infty$, $\sum_{n=0}^{\infty} \| G_2 ȕ_n - ȕ_n \| < \infty$

(ii) $\lim_{n\to\infty} ŧ_n < 1, \lim_{n\to\infty} đ_n < 1$

If $Ƭ_1$ and $Ƭ_2$ satisfy Condition $(Ă)$, then $\{ȕ_n\}$ converges-W to $\mathcal{CFP}$ of $Ƭ_1$ and $Ƭ_2$.

Proof: Let $ᵽ \in \Gamma$, we have

$$\begin{aligned} \|ȕ_{n+1} - ᵽ\|^2 &= \|đ_n\left[ɉ_n Ɇ_2 ȕ_n + (1 - ɉ_n) G_2 ȕ_n\right] + (1 - đ_n) ř_n - ᵽ\|^2 \\ &\le đ_n \|ɉ_n Ɇ_2 ȕ_n + (1 - ɉ_n) G_2 ȕ_n - ᵽ\|^2 + (1 - đ_n)\|ř_n - ᵽ\|^2 \qquad \textbf{(7)} \\ &\le đ_n [ɉ_n \|Ɇ_2 ȕ_n - ᵽ\|^2 + (1 - ɉ_n)\|G_2 ȕ_n - ᵽ\|^2 + (1 - đ_n)\|ř_n - ᵽ\|^2 \end{aligned}$$

$$= đ_n [ɉ_n \|Ɇ_2 ȕ_n - Ɇ_2 ᵽ\|^2 + (1 - ɉ_n)\|G_2 ȕ_n - G_2 ᵽ\|^2] + (1 - đ_n) d(ř_n, Ƭ_2 ᵽ)^2$$

$$\le đ_n [ɉ_n \|ȕ_n - ᵽ\|^2 + (1 - ɉ_n)(\varphi\|ȕ_n - ᵽ\|^2 + L \min\{\|ȕ_n - G_2 ᵽ\|^2, \|ᵽ - G_2 ᵽ\|^2\})$$
$$+ (1 - đ_n) H(Ƭ_2 ƶ_n, Ƭ_2 ᵽ)^2$$

$$\le đ_n [ɉ_n \|ȕ_n - ᵽ\|^2 + (1 - ɉ_n)\delta\|ȕ_n - ᵽ\|^2] + (1 - đ_n)\|ƶ_n - ᵽ\|^2$$

$$= đ_n \|ȕ_n - ᵽ\|^2 + (1 - đ_n)\|ƶ_n - ᵽ\|^2$$

And

$$\|ƶ_n - ᵽ\|^2 = \|ŧ_n[ƀ_n G_1 ȕ_n + (1 - ƀ_n) š_n] + (1 - ŧ_n) Ɇ_1 ȕ_n - ᵽ\|^2$$

$$= ŧ_n \|ƀ_n G_1 ȕ_n + (1 - ƀ_n) š_n - ᵽ\|^2 + (1 - ŧ_n)\|Ɇ_1 ȕ_n - ᵽ\|^2$$
$$- ŧ_n (1 - ŧ_n)\|ƀ_n G_1 ȕ_n + (1 - ƀ_n) š_n - Ɇ_1 ȕ_n\|^2$$

$$\le ŧ_n [ƀ_n \|G_1 ȕ_n - G_1 ᵽ\|^2 + (1 - ƀ_n)\|š_n - ᵽ\|^2] + (1 - ŧ_n)\|Ɇ_1 ȕ_n - Ɇ_1 ᵽ\|^2$$
$$- ŧ_n (1 - ŧ_n)[ƀ_n \|G_1 ȕ_n - š_n\|^2 + \|š_n - Ɇ_1 ȕ_n\|^2] \qquad \textbf{(8)}$$

$$\le ŧ_n [ƀ_n \left(\varphi\|ȕ_n - ᵽ\|^2 + L \min\{\|ȕ_n - G_1 ᵽ\|^2, \|ᵽ - G_1 ᵽ\|^2\}\right) + (1 - ƀ_n) d(š_n, Ƭ_1 ᵽ)^2]$$
$$+ (1 - ŧ_n)\|ȕ_n - ᵽ\|^2 - ŧ_n (1 - ŧ_n)[ƀ_n \|G_1 ȕ_n - š_n\|^2 + \|š_n - Ɇ_1 ȕ_n\|^2]$$

$\leq ŧ_n[ƀ_n\left(\delta||ŭ_n - ꝑ||^2 + L||ꝑ - Ǥ_1ꝑ||^2\right) + (1-ƀ_n)H(Ƭ_1ŭ_n, Ƭ_1ꝑ)^2]$
$\quad +\left(1-ŧ_n\right)||ŭ_n - ꝑ||^2 - ŧ_n\left(1-ŧ_n\right)[ƀ_n||Ǥ_1ŭ_n - š_n||^2 + ||š_n - ꞙ_1ŭ_n||^2]$
$= ||ŭ_n - ꝑ||^2 - ŧ_n\left(1-ŧ_n\right)[ƀ_n||Ǥ_1ŭ_n - š_n||^2 + ||š_n - ꞙ_1ŭ_n||^2]$

From **Equations 7 and 8,** we obtain

$$||ŭ_{n+1} - ꝑ||^2 \leq đ_n||ŭ_n - ꝑ||^2 + (1-đ_n)||ŭ_n - ꝑ||^2 \qquad \textbf{(9)}$$
$$-(1-đ_n)ŧ_n\left(1-ŧ_n\right)[ƀ_n||Ǥ_1ŭ_n - š_n||^2 + ||š_n - ꞙ_1ŭ_n||^2]$$
$$= ||ŭ_n - ꝑ||^2 - (1-đ_n)ŧ_n\left(1-ŧ_n\right)[ƀ_n||Ǥ_1ŭ_n - š_n||^2 + ||š_n - ꞙ_1ŭ_n||^2]$$

We have, $||ŭ_{n+1} - ꝑ||^2 \leq ||ŭ_n - ꝑ||^2$

Then, $\{ŭ_n\}$ is bounded and decreasing, also $\{ƶ_n\}$. Therefore, $\lim_{n\to\infty}||ŭ_n - ꝑ||$ exists, thus there exists $\{š_{n_k}\}$ of $\{š_n\}$ such that $š_{n_k} \rightharpoonup u$

To prove u is $\mathcal{CFP}$ of $Ƭ_1$ and $Ƭ_2$, by Lemma 1.10(ii), we get

$||ŭ_{n+1} - ꝑ||^2 = ||đ_n\left[ɉ_nꞙ_2ŭ_n + \left(1-ɉ_n\right)Ǥ_2ŭ_n\right] + \left(1-đ_n\right)ř_n - ꝑ||^2$
$= đ_n||ɉ_nꞙ_2ŭ_n + \left(1-ɉ_n\right)Ǥ_2ŭ_n - ꝑ||^2 + \left(1-đ_n\right)||ř_n - ꝑ||^2$
$\quad -đ_n\left(1-đ_n\right)||ɉ_nꞙ_2ŭ_n + \left(1-ɉ_n\right)Ǥ_2ŭ_n - ř_n||^2$
$\leq đ_n[ɉ_n||ꞙ_2ŭ_n - ꝑ||^2 + \left(1-ɉ_n\right)||Ǥ_2ŭ_n - ꝑ||^2] + \left(1-đ_n\right)d(ř_n, Ƭ_2ꝑ)^2$
$\quad -đ_n\left(1-đ_n\right)[ɉ_n||ꞙ_2ŭ_n - Ǥ_2ŭ_n||^2 + ||Ǥ_2ŭ_n - ř_n||^2]$
$\leq đ_n[ɉ_n||ꞙ_2ŭ_n - ꞙ_2ꝑ||^2 + \left(1-ɉ_n\right)||Ǥ_2ŭ_n - Ǥ_2ꝑ||^2] + \left(1-đ_n\right)H(Ƭ_2ƶ_n, Ƭ_2ꝑ)^2$
$\quad -đ_n\left(1-đ_n\right)[ɉ_n||ꞙ_2ŭ_n - Ǥ_2ŭ_n||^2 + ||Ǥ_2ŭ_n - ř_n||^2] \qquad$ **(10)**
$= đ_n||ŭ_n - ꝑ||^2 + \left(1-đ_n\right)||ƶ_n - ꝑ||^2 - đ_n\left(1-đ_n\right)[ɉ_n||ꞙ_2ŭ_n - Ǥ_2ŭ_n||^2 + ||Ǥ_2ŭ_n - ř_n||^2]$

From **Equations 8 and 10** we obtain

$$||ŭ_{n+1} - ꝑ||^2 \leq ||ŭ_n - ꝑ||^2 - \left(1-đ_n\right)ŧ_n\left(1-ŧ_n\right)[ƀ_n||Ǥ_1ŭ_n - š_n||^2 + ||š_n - ꞙ_1ŭ_n||^2]$$
$$-đ_n\left(1-đ_n\right)[ɉ_n||ꞙ_2ŭ_n - Ǥ_2ŭ_n||^2 + ||Ǥ_2ŭ_n - ř_n||^2]$$

Hence,

$\left(1-đ_n\right)ŧ_n\left(1-ŧ_n\right)[ƀ_n||Ǥ_1ŭ_n - š_n||^2 + ||š_n - ꞙ_1ŭ_n||^2]$
$+đ_n\left(1-đ_n\right)[ɉ_n||ꞙ_2ŭ_n - Ǥ_2ŭ_n||^2 + ||Ǥ_2ŭ_n - ř_n||^2] \leq ||ŭ_n - ꝑ||^2 - ||ŭ_{n+1} - ꝑ||^2$

Therefore, $\lim_{n\to\infty}||Ǥ_1ŭ_n - š_n|| = \lim_{n\to\infty}||š_n - ꞙ_1ŭ_n||$
$\quad = \lim_{n\to\infty}||ꞙ_2ŭ_n - Ǥ_2ŭ_n|| = \lim_{n\to\infty}||Ǥ_2ŭ_n - ř_n|| = 0 \qquad$ **(11)**

Therefore, $||ŭ_n - š_n|| \leq ||ŭ_n - Ǥ_1ŭ_n|| + ||Ǥ_1ŭ_n - š_n||$. Then $||ŭ_n - š_n|| \to 0$

Also, $||ƶ_n - ŭ_n|| = ||ŧ_n[ƀ_nǤ_1ŭ_n + (1-ƀ_n)š_n] + (1-ŧ_n)ꞙ_1ŭ_n - ŭ_n||$
$\leq ŧ_n[ƀ_n||Ǥ_1ŭ_n - ŭ_n|| + (1-ƀ_n)||š_n - ŭ_n||] + \left(1-ŧ_n\right)[||ꞙ_1ŭ_n - š_n|| + ||š_n - ŭ_n||]$

Then $||ƶ_n - ŭ_n|| \to 0$

Also,

$||ƶ_n - ř_n|| \leq ||ƶ_n - ŭ_n|| + ||ŭ_n - Ǥ_2ŭ_n|| + ||Ǥ_2ŭ_n - ř_n||$

Then $||ƶ_n - ř_n|| \to 0$

Now, because the sequence $\{ŭ_n\}$ is a bounded, there exists subsequence $\{ŭ_{n_k}\}$ of $\{ŭ_n\}$ such that $ŭ_{n_k} \rightharpoonup u$ for some $u \in Ě$, by Lemma 1.7, we have $u \in Ƭ_1u$. But $||ŭ_n - š_n|| \to 0$ then $||ŭ_{n_k} - š_{n_k}|| \to 0$ (i.e $š_{n_k} \rightharpoonup u$) hence $u \in Ƭ_1ŭ_n$. Again, by Lemma 1.7, we can show that

$u \in Ƭ_2 u$ but $||ƶ_n - ŭ_n|| \to 0$ then $||ƶ_{n_k} - ŭ_{n_k}|| \to 0$ (i.e $ƶ_{n_k} \rightharpoonup u$) hence $u \in Ƭ_2 ƶ_n$. Therefore, $u \in \mathcal{F}(Ƭ_1) \cap \mathcal{F}(Ƭ_2)$

4. Converges Strongly to Common Fixed Point

4.1.Theorem: Let $Ƭ_1, Ƭ_2 : Ĕ \longrightarrow Ǩ(Ĕ)$ be an HM-mapping, $ɟ_1, ɟ_2 : Ȟ \longrightarrow Ȟ$ are a N-mapping, and $Ǥ_1, Ǥ_2 : Ȟ \longrightarrow Ȟ$ are $(\varphi, L)^*$-W- contr mapping where φ is a $\mathcal{CF}$ defined by $\varphi(ҳ) = \deltaҳ, 0 < \delta \leq 1$ such that $\Gamma \coloneqq (\bigcap_{i=1}^{2} \mathcal{F}(Ƭ_i)) \cap (\bigcap_{i=1}^{2} \mathcal{F}(ɟ_i)) \cap (\bigcap_{i=1}^{2} \mathcal{F}(Ǥ_i)) \neq \emptyset$. Let $\{ŭ_n\}$ be generated by:

$$\begin{cases} ŭ_1 \in Ĕ\,, \; Ĕ_1 = Ĕ \\ ƶ_n \in ŧ_n[ƀ_n Ǥ_1 ŭ_n + (1 - ƀ_n) Ƭ_1 ŭ_n] + (1 - ŧ_n) ɟ_1 ŭ_n \\ ۷_n \in đ_n \left[ɉ_n ɟ_2 ŭ_n + (1 - ɉ_n) Ǥ_2 ŭ_n \right] + (1 - đ_n) Ƭ_2 ƶ_n \\ Ĕ_{n+1} = \{ƶ \in Ĕ_n : ||۷_n - ƶ|| \leq ||ŭ_n - ƶ||\} \\ ŭ_{n+1} = P_{Ĕ_{n+1}} ŭ_0\,, \; \forall n \geq 1 \end{cases} \tag{12}$$

For all $n \geq 1$ where $\{ŧ_n\}$, $\{ƀ_n\}$, $\{đ_n\}$ and $\{ɉ_n\} \subset (0,1]$. Assume that

(i) $\sum_{n=0}^{\infty} || Ǥ_1 ŭ_n - ŭ_n|| < \infty$, $\sum_{n=0}^{\infty} || Ǥ_2 ŭ_n - ŭ_n|| < \infty$

(ii) $\lim_{n\to\infty} ŧ_n < 1, \lim_{n\to\infty} đ_n < 1$

If $Ƭ_1$ and $Ƭ_2$ satisfy Condition $(Ă)$, then

- $Ĕ_n$ is an nonempty closed convex.
- $\Gamma \subseteq Ĕ_n$ for each $n \geq 1$.
- The sequence converges-S to $\mathcal{CFP}$ of $Ƭ_1$ and $Ƭ_2$
- $\lim_{n\to\infty} \left\| ŭ_n - š_n \right\| = \lim_{n\to\infty} \left\| ƶ_n - ř_n \right\| = 0$
- $u = P_\Gamma ŭ_0$

Proof: Following the same proof method above, we get what is required

4.2. Theorem: Let $Ƭ_1, Ƭ_2 : Ĕ \longrightarrow Ǩ(Ĕ)$ be an HM-mapping and $ɟ_1, ɟ_2 : Ȟ \longrightarrow Ȟ$ are an N-mapping, $Ǥ_1, Ǥ_2 : Ȟ \longrightarrow Ȟ$ are $\varphi, L)^*$-W- contr mapping where φ is a $\mathcal{CF}$ defined by $\varphi(ҳ) = \deltaҳ, 0 < \delta \leq 1$ such that $\Gamma \coloneqq (\bigcap_{i=1}^{2} \mathcal{F}(Ƭ_i)) \cap (\bigcap_{i=1}^{2} \mathcal{F}(ɟ_i)) \cap (\bigcap_{i=1}^{2} \mathcal{F}(Ǥ_i)) \neq \emptyset$. Let $\{ŭ_n\}$ be generated by:

$$\begin{cases} ŭ_1 \in Ĕ \text{ chosen arbitrarily} \\ ƶ_n \in ŧ_n[ƀ_n Ǥ_1 ŭ_n + (1 - ƀ_n) Ƭ_1 ŭ_n] + (1 - ŧ_n) ɟ_1 ŭ_n \\ ۷_n \in đ_n \left[ɉ_n ɟ_2 ŭ_n + (1 - ɉ_n) Ǥ_2 ŭ_n \right] + (1 - đ_n) Ƭ_2 ƶ_n \\ Ĕ_n = \{ƶ \in Ĕ : ||۷_n - ƶ|| \leq ||ŭ_n - ƶ||\} \\ Ň_n = \{ƶ \in Ĕ : \langle ŭ_0 - ŭ_n, ŭ_n - ƶ \rangle \geq 0\} \\ ŭ_{n+1} = P_{Ĕ_n \cap Ň_n} ŭ_0\,, \; \forall n \geq 1 \end{cases} \tag{13}$$

For all $n \geq 1$ where $\{ŧ_n\}$, $\{ƀ_n\}$, $\{đ_n\}$ and $\{ɉ_n\} \subset (0,1]$. Assume that

(i) $\sum_{n=0}^{\infty} || Ǥ_1 ŭ_n - ŭ_n|| < \infty$, $\sum_{n=0}^{\infty} || Ǥ_2 ŭ_n - ŭ_n|| < \infty$

(ii) $\lim_{n\to\infty} ŧ_n < 1, \lim_{n\to\infty} đ_n < 1$

If $Ƭ_1$ and $Ƭ_2$ satisfy Condition (Ă) then

- $\check{E}_n$ and $\check{N}_n$ are nonempty closed and convex
- $\Gamma \subseteq \check{E}_n \cap \check{N}_n$ for each $n \geq 1$.
- The sequence converges-S to $\mathcal{CFP}$ of $Ƭ_1$ and $Ƭ_2$
- $\lim_{n\to\infty} \left\|\left|ȕ_n - \check{s}_n\right|\right\| = \lim_{n\to\infty} \left\|\left|ƶ_n - \check{r}_n\right|\right\| = 0$
- $u = P_\Gamma ȕ_0$

Proof: Following the same proof method above, we get what is required

5. Conclusion

In this study, we introduced the concept of $(\varphi, L)^*$-weak contraction mapping which is a generalization of the concept (φ, L)-weak contraction mapping. New iterative techniques in $\check{H}$ are introduced, convergence-S and convergence-W theorems to $\mathcal{CFP}$ via HM mapping, nonexpansive single-valued mappings, and $(\varphi, L)^*$-weak contr mappings are proved.

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Conflict of Interest

The authors declare that they have no conflicts of interest.

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