



Z –Essentail Small Compressible Modules

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Received:12/November/2025

Accepted: 23/February/2026

Published: 20/July/2026

doi.org/10.30526/39.3.4325



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Abstract

Let C be the left unitary R -module, and let R be a commutative ring with 1. In this paper, we introduce a detail and study the concept of a Z -essential small compressible module (Z -ess- s -comp module for short), if C can be embedded in each of its non-zero Z -small sub-module. Equivalently, C is a Z -ess- s -comp module if there exists a monomorphism from C into N whenever $0 \neq N \ll_{Z\text{-ess}} C$, as a generalization of the compressible module (comp module for short), and gives some of their properties, characterizations, and examples. On the other hand, we study the relations between Z -ess- s -compressible modules and some classes of modules. We also presented the two concepts of Z -essential small prime (module, submodule) Z -essential small uniform module (where C is called a Z -essential small uniform module if any non-zero Z -essential small sub-module of C is essential. and provide the relationships between Z -ess- s -compressible modules, Z -essential small prime module, and Z -essential small uniform module.

Keywords: Z -small sub-module, Z -essential small sub-module, compressible module, Z -essential small compressible module.

1. Introduction

Let C be the left unitary R -module, and let R be a commutative ring with 1. The authors introduced and examined small sub-modules of the notion. If every K sub-module of C with $N + K = C$ implies that $K = C$, then a sub-module N of an R -module C is called a small sub-module ($N \ll C$)¹⁻⁷. Introducing the concept of Z -small sub-modules, N is said to be a Z -small sub-module in M (briefly $N \ll_z C$), where $N \leq M$, if $N + B = C$ with $B \supseteq Z_2(C)$ and $B \leq M$, then $B = C$ ⁸ (where the second singular sub-module $Z_2(C)$ of C is the sub-module of C containing $Z(C)$, such that $\frac{Z_2(C)}{Z(C)}$ is the singular sub-module of $\frac{C}{Z(C)}$. For more information about $Z_2(C)$)^{9,10}.

The concept of Z -essential small sub-modules was introduced and studied by¹¹ A sub-module N of a module C is called Z -essential small sub-module (briefly $N \ll_{Z\text{-ess}} C$ if $N + K = C$, $K \leq_{Z\text{-ess}} C$, implies $K = C$ (where a sub-module K of C is called Z -essential (briefly $K \leq_{Z\text{-ess}} C$) if whenever $K \cap B = 0$, $K \subseteq Z_2(C)$ yields $B = (0)$.

In 1976, the term comp module was coined. If an R -module C can be embedded in every non-zero sub-module, that is, if there is a monomorphism $f: C \rightarrow N$ for any N that is a non-zero sub-module N of C , then C is said to be comp in⁶, introduced and studied Z -small comp where (If C embedded in each of its nonzero Z -small sub-module, then R -module C is called Z -small comp, equivalently, C is Z -small comp if there is a monomorphism $f: C \rightarrow N$ whenever $0 \neq N \ll_z C$). For more on this topic¹²⁻¹⁶.

Introducing a Z -small prime module and Z -small prime sub-module where C is called Z -small prime if $\text{ann } C = \text{ann } N$, for each nonzero Z -small sub-mod N of C in⁸. And a sub-module N

of an R -module is Z -small prime if whenever $r \in R$, $x \in C$ with $(x) \ll_Z C$ and $rx \in N$ implies either $x \in N$ or $r \in [N : C]$. For more on this topic¹⁷⁻¹⁹.

As an extension of Z -small uniform, Z -ess-tail small uniform (Z -ess- s -uniform module for short) was developed in⁸. An R -module C is called Z -small uniform if every nonzero Z -small sub-module of C is essential. And a R -module C is called Z -ess-tail small uniform (Z -ess- s -uniform) if for any non zero Z -ess-tail small sub-module of C is essential, for more information²⁰⁻²³.

In this paper, as a generalization of the Z -small comp modules, we introduce the basic concept of the Z -ess- s -com modules and examine some of its properties and applications. We will also explore the relationship between the advantages of the Z -ess- s -comp, the Z -small comp modules, the small comp modules, and the comp modules, providing examples and characteristics. We also provide the relationships between Z -ess- s -comp modules, Z -ess- s -prime, and Z -ess- s -uniform.

2. Preliminaries

2.1. Definition:

- Let N be a sub-module of C . If $N + B = C$ with $B \supseteq Z_2(C)$ and $B \leq C$, then $B = C$. This means that N is a Z -small sub-module in C (briefly $N \ll_Z C$)⁸.
- A sub-module N of a module C is called Z -ess-tail small sub-module (briefly $N \ll_{Z-ess} C$) if $N + K = C$, $K \leq_{Z-ess} C$, implies $K = C$ ¹¹.
- If C embedded in each of its nonzero sub-module. That is for each non-zero sub-module N of $C \exists$ a monomorphism $f : C \rightarrow N$. then R -module C is called comp¹².
- If C embedded in each of its nonzero Z -small sub-module, then R -module C is called a Z -small comp equivalently, C is Z -small comp if $\exists$ a monomorphism $f : C \rightarrow N$ whenever $0 \neq N \ll_Z C$.

2.2. Remark

- In generally, every small sub-module is Z -essential small, but the converse is not true. For instance, in the Z -module Z_{24} . Only $\langle \bar{2} \rangle, \langle \bar{4} \rangle, Z_{24}$ are Z -ess-tail in Z_{24} . Note that $\langle \bar{2} \rangle \ll_{Z-ess} Z_{24}$ since $\langle \bar{2} \rangle + \langle \bar{2} \rangle \neq Z_{24}$, we have $\langle \bar{2} \rangle \ll_{Z-ess} Z_{24}$. On the other hand $\langle \bar{2} \rangle$ is not small in Z_{24} .
- Every Z -ess-tail small sub-module is an essential small sub-module.
- Every Z -ess-tail small sub-module is a Z -small sub-module.
- Every small sub-module is a Z -small essential sub-module¹¹.

2.3. Lemma

Let C be an R -module. The following assertions hold¹¹:

- let $N \leq C$ if $N \ll_{Z-ess} C$, then $N \ll_Z C$.
- If $K \ll_{Z-ess} N \leq C$, then $K \ll_{Z-ess} C$.
- If $f : C_1 \rightarrow C_2$ is an epimorphism and $N \ll_{Z-ess} C_1$, then $f(N) \ll_{Z-ess} C_2$.
- If $K \leq N \leq C$ and $N \ll_{Z-ess} C$, then $K \ll_{Z-ess} C$.

2.4. Lemma

- If $Z_2(C) = C$, $N \leq C$, then the following assertions are identical¹¹:

$$N \ll C, \quad N \ll_{ess} C, \quad N \ll_{Z-ess} C$$

- Let C be a Z -uniform module (i.e. every nonzero sub-module of C is Z -essential)²⁴, $N \leq C$. Then, $N \ll C$ if and only if $N \ll_{Z-ess} C$.

2.5. Lemma

Let C_1 and C_2 be an R -module, $N_1 \leq C_1$, $N_2 \leq C_2$. Then, $N_1 \ll_{Z-ess} C_1$, $N_2 \ll_{Z-ess} C_2$ if and only if $N_1 \oplus N_2 \ll_{Z-ess} C_1 \oplus C_2$.¹¹

3. Z –Essential Small Compressible Modules

In the section, a generalization of a Z – ess – s – compressible module was given and studied.

3.1. Definition:

An R -module C is said to be Z –essential small compressible (Z – ess – s – comp module for short), if C can be embedded in each of its non zero Z –essential small sub–modules. Equivalently, C is Z – ess – s – comp if there exists a monomorphism $f: C \rightarrow N$ whenever $N \ll_{Z-ess} C$.

3.2. Example:

- Since every sub–module of the Z -module $Z_{p\infty}$ is Z - essential small, so $Z_{p\infty}$ is not Z – ess – s – compressible since it can not be embedded in any Z - essential small sub–module.
- Z_{12} as Z -module is not Z -small comp, since $(\bar{2}) \ll_Z Z_{12}$ but there is no one to one homomorphism between Z_{12} and $(\bar{2})$. Also Z_{12} as Z -module is not Z – ess – s –comp since $(\bar{2}) \ll_{Zes} Z_{12}$ also there is no one–to–one homomorphism between them.
- Z as Z -module is Z – ess – s –comp (by logic) since it is comp.
- Every simple module is Z – ess – s – comp but not conversely, since Z as a Z -module is Z – ess – s – comp but not conversely.
- Z_4 as Z_2 -module is not Z – ess – s – comp.

3.3. Proposition:

A module C is Z – ess – s – comp if and only if C can be embedded in Rx for each non zero element x in C and $Rx \ll_{Z-ess} C$.

Proof: ($\Rightarrow$) by definition (3.1)

($\Leftarrow$) Let $0 \neq N \leq C$ and let $0 \neq x \in N$, so by assumption $Rx \ll_{Z-ess} C$, and there is a monomorphism $f: C \rightarrow Rx$. Thus, the composition $iof: C \rightarrow N$ is a monomorphism where $i: Rx \rightarrow N$ is the inclusion homomorphism. Therefore, C is a Z – ess – s – comp.

3.4. Remarks:

- Every Z – ess – s – comp is a small comp.

Proof:

Suppose that C is a Z – ess – s – comp and let N be a small sub–module of C . By Remark (2.2), N is Z –essential small and thus by assumption C can be embedded in N .

But the converse is not true: for example, Z_6 as a Z -module is a small comp but not Z – ess – s –comp since there is no non zero Z –essential in Z_6 so by logic every sub–module of Z_6 is Z –essential small. But Z_6 cannot be embedded in each of them.

- Every essential small comp is Z – ess – s – comp.

Proof: Suppose that C is an essential small comp and let N be Z – essential small sub–module of C . By Remark (2.2) N is essential small, so by assumption C can be embedded in N . Therefore C is Z – ess – s – comp.

But the converse is not true: for example every sub–module Z_6 as Z_6 is a Z –essential $(\bar{0}), (\bar{2}), (\bar{3})$. To check if they are Z –essential small or not. $(\bar{2}), (\bar{3})$ are not Z –essential small since $(\bar{2}) + (\bar{3}) = Z_6$, $(\bar{2}), (\bar{3}) \leq_{Z-ess} Z_6$ but $(\bar{2}), (\bar{3}) \neq Z_6$, so $(\bar{2}), (\bar{3})$ are not Z –essential small. Then, Z_6 as Z_6 is a Z – ess – s – comp (by logic) since it has no Z –essential small sub–module. But Z_6 as a Z_6 - module is not essential small comp since every sub–module of Z_6 is essential small, but Z_6 can not be embedded in every non zero essential small.

- Every Z – small comp R –module is Z – ess – s – comp.

Proof: Since every Z –essential small sub–module is Z -small⁶ so the statement is true.

The converse of a previous is not true in general: since if C is Z – ess – s – comp. If $N \ll_Z C$ then may be not Z –essential small.

- Every comp module is Z – ess – s – comp.

Proof: clear.

The converse of the above is not true in general since Z_6 as Z_6 is Z -ess- s -comp (by logic) but it is not comp.

We can summarize the above relation by the following **Figure 1**:

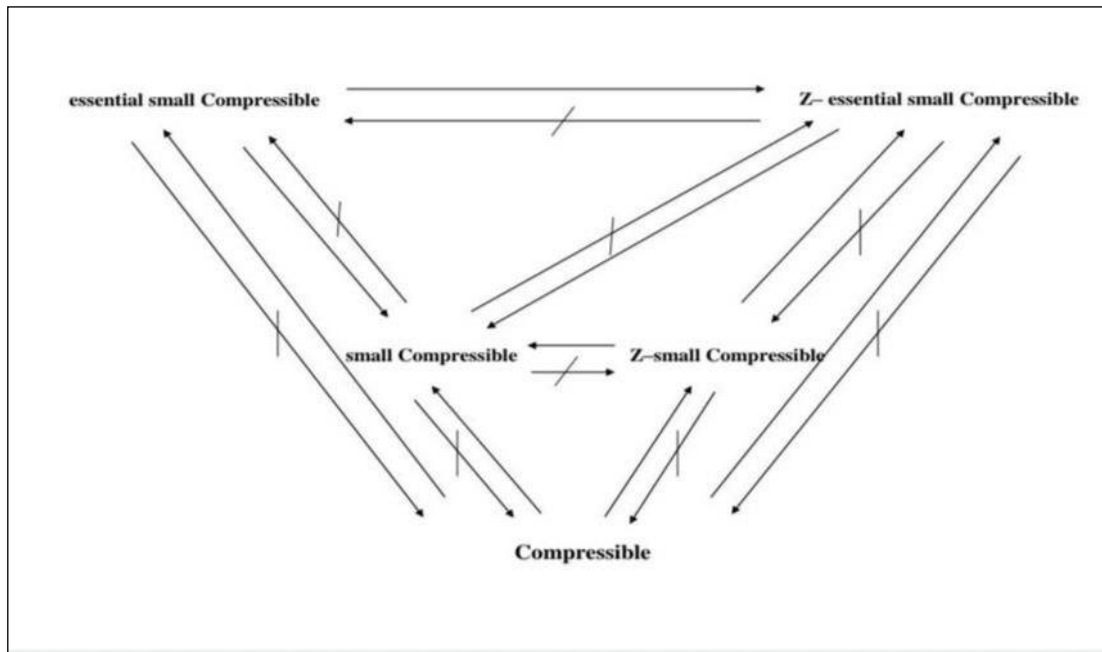


Figure 1: relations between Z -ess- s -comp and other kinds of modules

In the following proposition, we show that the Z -ess- s -comp and comp modules are equivalent under certain conditions

3.5. Propositions:

- Let C be a Z -ess- s -comp module, such that every submodule of C contains a non zero Z -essential small sub-module of C , then C is comp.

Proof: Let $0 \neq N \leq C$, by assumption N contains a nonzero sub-module K such that $K \ll_{Z\text{-ess}} N$. By lemma (2.3) we have $K \ll_{Z\text{-ess}} C$. But C is Z -ess- s -comp, so there exists a monomorphism $f: C \rightarrow K$ and, hence $i \circ f: C \rightarrow N$ is a monomorphism. Therefore, C is comp.

- A Z -ess- s -comp module is comp if every cyclic sub-module of C is Z -essential small in C .

Proof: Let $0 \neq N \leq C$ and let $0 \neq a \in N$, so by assumption $Ra \ll_{Z\text{-ess}} C$ and so there is a monomorphism $f: C \rightarrow Ra$ and hence $i \circ f: C \rightarrow N$ is a monomorphism. Therefore C is a comp.

3.6. Proposition:

A Z -essential small sub-module of a Z -ess- s -comp module C is also Z -ess- s -comp.

Proof: Let $0 \neq H \ll_{Z\text{-ess}} C$ and let $0 \neq K \ll_{Z\text{-ess}} N$. Then by lemma (2.3), $K \ll_{Z\text{-ess}} C$. But C is Z -ess- s -comp, so there exists a monomorphism $L: C \rightarrow K$ and hence $L \circ i: H \rightarrow K$ is a monomorphism where $i: H \rightarrow C$ is the inclusion homomorphism. Hence H is Z -ess- s -comp.

3.7. Remark:

A homomorphic image of a Z -ess- s -comp module need not be a Z -ess- s -comp in general. For example, Z as a Z -module is Z -ess- s -comp but $\frac{Z}{12Z} \cong Z_{12}$ is not Z -ess- s -comp.

3.8. Proposition:

A direct summand of a $Z - \text{ess} - s - \text{comp}$ module is also $Z - \text{ess} - s - \text{comp}$.

Proof: Let $C = L \oplus K$ be a $Z - \text{ess} - s - \text{comp}$ module and let $0 \neq A \ll_{Z-\text{ess}} L$. Thus, by lemma (2.5) then $A \oplus 0 \ll_{Z-\text{ess}} C$. By assumption there is a homomorphism f which one-to-one from C to $A \oplus 0$. Its clear $A \oplus 0 \simeq A$, so $f: C \rightarrow A$ is mono-morphism. Thus, $L \xrightarrow{j_L} C \xrightarrow{f} A$ is a monomorphism where j_L is the injection of L in to C . Hence, L is a $Z - \text{ess} - s - \text{comp}$.

3.9. Corollary:

Let C be a semisimple R -module. If C is $Z - \text{ess} - s - \text{comp}$, then every nonzero sub-module of C is $Z - \text{ess} - s - \text{comp}$.

Proof: Suppose that C is $Z - \text{ess} - s - \text{comp}$. Let $0 \neq N \leq C$, so by assumption $N \leq^\oplus C$. Thus, by proposition (3.8), so N is $Z - \text{ess} - s - \text{comp}$.

3.10. Proposition:

If $C_1 \simeq C_2$ (C_1 and C_2 are two R -modules). Then C_1 is a $Z - \text{ess} - s - \text{comp}$ if and only if C_2 is a $Z - \text{ess} - s - \text{comp}$.

Proof: Suppose that C_1 is $Z - \text{ess} - s - \text{comp}$. Let $\psi: C_1 \rightarrow C_2$ be an isomorphism, $\psi^{-1}: C_2 \rightarrow C_1$ is well-be homomorphism. Let $N \ll_{Z-\text{ess}} C_2$, then $\psi^{-1}(N) \ll_{Z-\text{ess}} C_1$ by lemma (2.3). Put $L = \psi^{-1}(N)$, $g: C_1 \rightarrow L$ is a monomorphism and $f = \psi|_L$ then $f: L \rightarrow C_2$ is a monomorphism. $f(L) = \psi\psi^{-1}(N) = N$, hence, $f: L \rightarrow N$ is a monomorphism. Now, we have the composition $h = f \circ g \circ \psi^{-1}: C_2 \xrightarrow{\psi^{-1}} C_1 \xrightarrow{g} L \xrightarrow{f} N$ is a monomorphism. Thus, C_2 is $Z - \text{ess} - s - \text{comp}$.

3.11. Proposition:

Let $C = C_1 \oplus C_2$ be an R -module such that $\text{ann } C_1 \oplus \text{ann } C_2 = R$. Then, C is $Z - \text{ess} - s - \text{comp}$ if and only if C_1 and C_2 are $Z - \text{ess} - s - \text{comp}$.

Proof: ($\Rightarrow$) its follow from proposition (3.8).

($\Leftarrow$) Let $0 \neq N \ll_{Z-\text{ess}} C$. Then by²⁵, $N = K_1 \oplus K_2$, for some $0 \neq K_1 \leq C_1 \leq C$ and $0 \neq K_2 \leq C_2 \leq C$. Since $N \ll_{Z-\text{ess}} C$ so by lemma (2.5) we have $K_1 \ll_{Z-\text{ess}} C_1$ and $K_2 \ll_{Z-\text{ess}} C_2$. But by assumption C_1 and C_2 are $Z - \text{ess} - s - \text{comp}$, so there is monomorphism, $f: C_1 \rightarrow K_1$ and $g: C_2 \rightarrow K_2$. Define $h: C \rightarrow N$ by $h(a, b) = (f(a), g(a))$, it is clear that h is a monomorphism, and hence C is $Z - \text{ess} - s - \text{comp}$.

3.12. Corollary:

Let $\{C_i\}_{i=1}^n$ be a finite family of a $Z - \text{ess} - s - \text{comp}$ R -module such that $\sum_{i=1}^n \text{ann } C_i = R$. Then, $\bigoplus_{i=1}^n C_i$ is also $Z - \text{ess} - s - \text{comp}$.

Proof: By induction using previous proposition.

3.13. Proposition:

If $Z_2(C) = C$, then, the following are equivalent:

- * Small comp.
- ** Essintial small comp.
- *** $Z - \text{ess} - s - \text{comp}$.

Proof: (*) $\Rightarrow$ (**) Let $0 \neq N$ be a sub-module of small comp module C such that $N \ll_{\text{ess}} C$. Thus, by assumption $N \ll C$. So C can be embedded in N because C is a small comp. Therefore, C is essential small comp.

(**) $\Rightarrow$ (*) by¹⁴ Remark (3.3, every essential small comp module is a small comp.

(**) $\Rightarrow$ (***) by Remark (3.4), every essential small comp module is $Z - \text{ess} - s - \text{comp}$.

(***) $\Rightarrow$ (**) let N be nonzero essential small in a module C thus by assumption $N \ll_{Z-\text{ess}} C$ and so there is a monomorphism $f: C \rightarrow N$. Therefore C is an essential small comp.

(***) $\Rightarrow$ (*) by remark (3.4), every $Z - \text{ess} - s - \text{comp}$ module is a small comp.

(*) $\Rightarrow$ (***) Let C be a small comp and let $0 \neq N \ll_{Z-\text{ess}} C$ by remarks (2.3) $N \ll C$. But C is small comp so there exist monomorphism, $f: C \rightarrow N$. Thus C is $Z - \text{ess} - s - \text{comp}$ module.

Now, we have the following:

3.14. Proposition:

If C is a Z -uniform module. Then, C is a small comp if and only if C is Z - ess - s - comp.

Proof: Clear from lemma (2.4)

4. Z -essential small uniform modules

In this section, we give a Z -essential small uniform and some remarks and examples before giving a new conclusion, we need the following concepts and propositions:

4.1. Definition:

Let C be an R -module, C is called a Z -essential small uniform (Z - ess - s -uniform) if for any nonzero Z -essential small sub-module of C is essential.

4.2. Remarks and Examples:

- It is clear that every uniform module is Z - ess - s -uniform. but not converse, for example Z_6 as Z_6 -module is Z - ess - s -uniform (by logic since the only nonzero sub-modules of Z_6 are $\langle \bar{2} \rangle, \langle \bar{3} \rangle$, but $\langle \bar{2} \rangle, \langle \bar{3} \rangle \ll_{Z-ess} Z_6$) while Z_6 is not uniform as Z_6 -module.
- Every Z -uniform module is Z - ess - s -uniform (where a module C is a Z -uniform if every non zero sub-module of C is essential²²).

Proof: Let $0 \neq N \ll_{Z-ess} C$ so by Remark (2.2), $N \ll_Z C$ and since C is Z -uniform so $N \leq_e C$. Therefore, C is a Z - ess - s -uniform.

- Z_4 as Z -module is a Z - ess - s -uniform (since it is uniform)

4.3. Definition:

Let C be an R -module and $N < C$. N is called a Z -essential small prime sub-module (Z - ess - s -prime) if $rx \in N$ with $\langle x \rangle \ll_{Z-ess} C$ ($r \in R, x \in C$ implies either $x \in N$ or $r \in [N:C]$).

4.4. Remarks and Examples:

- Every prime sub-module is a Z - ess - s - prime sub-module.

Proof: clear.

- Every Z - ess - s -prime sub-module is a small prime sub-module.

Proof: Let $0 \neq N \leq C$ be a Z - ess - s -prime such that $rx \in N$ with $\langle x \rangle \ll C$ (where $r \in R, x \in C$) by remark (2.2) $\langle x \rangle \ll_{Z-ess} C$, by assumption either $x \in N$ or $r \in [N:C]$.

- Every Z - small prime sub-module is a Z - ess - s -prime sub-module.

Proof: Let $0 \neq N < C$ be Z - small prime sub-module of an R -module C such that $rx \in N$ with $\langle x \rangle \ll_{Z-ess} C$ (where $r \in R, x \in C$) by remark (2.2) $\langle x \rangle \ll_Z C$, by assumption either $x \in N$ or $r \in [N:C]$.

4.5. Definition:

An R -module C is called a Z -essential small prime module (Z - ess - s -prime) if $ann C = ann N$ for every non zero Z -essential small sub-module N of C .

4.6. Remarks and Examples:

- Every prime module is a Z - ess - s - prime.

Proof: clear.

- Every Z - ess - s -prime module is small prime.

Proof: By remark (2.2), we get result.

The converse is not true for example: Z_6 as a Z -module is a small prime but not Z -essential small prime module since $\langle \bar{3} \rangle \ll_{Z-ess} Z_6$ but $ann(Z_6) = 6Z \neq ann\langle \bar{3} \rangle = 2Z$.

- Every Z - small prime module is a Z - ess - s - prime module.

Proof: By Remark (2.2), we get result.

- Every semi simple is a Z -essential small prime, but the converse is not true for example: Z as a Z -module is Z -essential small prime but not semi simple.

4.7. Proposition:

If C is a Z - ess - s - comp R -module, then, C is Z - ess - s - prime and Z - ess - s -uniform.

Proof: Let $0 \neq N$ be a Z -essential small sub-module of C to show that C is a Z -ess- s -prime. Let $I = \text{ann}_R N$, thus $IN = (0)$. But C is Z -ess- s -comp so there exists a monomorphism $f: C \rightarrow N$. Notice that, $f(IC) = If(C) \subseteq IN = (0)$; this means that $I \subseteq \text{ann}_R C$, thus $\text{ann}_R C = \text{ann}_R N$ i.e. C Z -ess- s -prime.

To show that C is Z -ess- s -uniform, let $0 \neq x \in N$, so by lemma (2.3) $Rx \ll_{Z\text{-ess}} C$. Since C is Z -ess- s -comp, there exists a monomorphism $f: C \rightarrow Rx$. So $f(x) = rx \neq 0$, for some $r \in R, r \neq 0$. Take $0 \neq a \in C$ and $f(a) = tx$, for some $t \in R$. Since f is monomorphism, $tx \neq 0$. We have $f(ra) = rf(a) = rtx = trx = tf(x) = f(tx)$. As f is monomorphism, we have $ra = tx$. But $tx \neq 0$, thus $ra \neq 0$ and $ra \in N$, this means that $N \leq_e C$ and, therefore, C is Z -ess- s -uniform.

For the converse of last proposition, we give the following theorem as a partial answer.

4.8. Theorem:

Let $C = Rm_1 \oplus Rm_2 \oplus \dots \oplus Rm_k$, where $m_1, m_2, \dots, m_k \in C$. If C is a Z -ess- s -uniform and a Z -ess- s -prime, then C is a Z -ess- s -comp module.

Proof: Let $C = Rm_1 \oplus Rm_2 \oplus \dots \oplus Rm_k$, where $m_1, m_2, \dots, m_k \in C$, let $N \ll_{Z\text{-ess}} C, N \neq 0$. By assumption C is Z -ess- s -uniform so $N \leq_e C$. Thus for $m_i \in C$ ($i = 1, 2, \dots, k$), $\exists t_i \in R - \{0\}$ such that $0 \neq t_i m_i \in N$. Put $t = t_1, \dots, t_k$.

Claim $t \neq 0$. Assume $t = 0$, hence $t_1, \dots, t_k m_k = 0$, thus $(t_1, \dots, t_{k-1})(t_k m_k) = 0$. But $\langle t_k m_k \rangle \leq N \ll_{Z\text{-ess}} C$, so by lemma (2.3) $\langle t_k m_k \rangle \ll_{Z\text{-ess}} C$ and since C is Z -ess- s -prime, we get $t_1, \dots, t_{k-1} \in \text{ann } C$. Then, $(t_1, \dots, t_{k-1})m_k = \langle 0 \rangle$, hence $(t_1 t_2, \dots, t_{k-2})(t_{k-1} m_{k-1}) = 0$. This implies $t_1 t_2, \dots, t_{k-1} \in \text{ann } C$. Since $\langle t_{k-1} m_{k-1} \rangle \ll_{Z\text{-ess}} C$ and C is Z -ess- s -prime. Thus, $t_1 t_2, \dots, t_{k-2} m_{k-2} = 0$. By repeating this process we get $t_1 m_1 = 0$ hence, which is a contradiction. Therefore, $t = 0$. Now, define $f: C \rightarrow N$ by $f(m) = tm$, for each $m \in C$. Clearly $tm \in N$, for each m .

$\ker f = \langle 0 \rangle$ since: assume $f(m) = \langle 0 \rangle$ for some $m \in C$. Since $m = r_1 m_1 + \dots + r_k m_k$ for some $r_i \in R, \forall i = 1, \dots, k$. Hence, $tr_i m_i = 0$, for each $i = 1, \dots, k$. But $tr_1 m_1 = 0$, so we get $(t_1 t_2, \dots, t_k)(r_1 m_1) = 0$ and hence $(t_2 t_3, \dots, t_k r_1)(t_1 m_1) = \langle 0 \rangle$ since $\langle t_1 m_1 \rangle \ll_{Z\text{-ess}} N$ and C is Z -ess- s -prime, we get $t_2, \dots, t_k r_1 \in \text{ann } C$ and so $(t_2, \dots, t_k r_1)(m_2) = 0$. Thus, by a similar argument, we get $t_3, \dots, t_k r_1 \in \text{ann } M$. Repeating this process, we have $r_1 \in \text{ann } C$. Therefore, $r_1 m_1 = 0$, similarly $r_i m_i = 0, i = 2, \dots, k$. Therefore $m = 0$, that is $\ker f = \langle 0 \rangle$ and hence C is a Z -ess- s -comp module.

5. Conclusion

In this paper, we succeeded in presenting the concepts of Z -ess- s -comp, and we showed that it is an expansion of the concept of Z -small comp module, and we presented some important relations between Z -ess- s -comp modules and other kinds of modules. We also presented the two concepts Z -ess- s -prime, and a Z -ess- s -uniform module, and provided the relationships between the Z -ess- s -comp modules, Z -ess- s -prime module, and Z -ess- s -uniform module.

Conflict of Interest

There is no conflict of interest

Funding

None

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