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Z −Essentail Small Compressible Modules

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Abstract

Let C be the left unitary R–$module$, and let R be a commutative ring with 1. In this paper, we introduce a detail and study the concept of a Z-essential small compressible module ($Z-ess-s$ −comp module for short), if C can be embedded in each of its non-zero Z-small sub−module. Equivalently, C is a $Z-ess-s$ −comp module if there exists a monomorphism from C into N whenever $0 \neq N \ll_{Z-ess} C$, as a generalization of the compressible module (comp module for short), and gives some of their properties, characterizations, and examples. On the other hand, we study the relations between $Z-ess-s$ −compressible modules and some classes of modules. We also presented the two concepts of Z −essentail small prime (module, submodule) Z −essential small uniform module (where C is called a Z −essentail small uniform module if any non-zero Z −essential small sub−module of C is essential. and provide the relationships between $Z-ess-s$ −compressible modules, Z −essentail small prime module, and Z −essential small uniform module.

Keywords: Z-small sub−module, Z- essential small sub−module, compressible module, Z- essential small compressible module.

1. Introduction

Let C be the left unitary $R-module$, and let R be a commutative ring with 1. The authors introduced and examined small sub-modules of the notion. If every K sub-module of C with $N+K=C$ implies that $K=C$, then a sub-module N of an R-module C is called a small sub-module ($N \ll C$)[1-7]. Introducing the concept of $Z-$ small sub−modules, N is said to be a Z-small sub−module in M (briefly $N \ll_z C$), where $N \leq M$, if $N+B=C$ with $B \supseteq Z_2(C)$ and $B \leq M$, then $B=C$[8] (where the second singular sub-module $Z_2(C)$ of C is the sub−module of C containing $Z(C)$, such that $\frac{Z_2(C)}{Z(C)}$ is the singular sub−module of $\frac{C}{Z(C)}$. For more information about $Z_2(C)$[9,10].

The concept of Z- essential small sub−modules was introduced and studied by[11] A sub−module N of a module C is called Z −essential small sub−module (briefly $N \ll_{Z-ess} C$ if $N+K=C$, $K \leq_{Z-ess} C$, implies $K=C$ (where a sub−module K of C is called Z −essential (briefly $K \leq_{Z-ess} C$) if whenever $K \cap B=0$, $K \subseteq Z_2(C)$ yields $B=(0)$.

In 1976, the term comp module was coined. If an R −module C can be embedded in every non-zero sub−module, that is, if there is a monomorphism $f: C \longrightarrow N$ for any N that is a non-zero sub−module N of C, then C is said to be comp in [6], introduced and studied Z-small comp where (If C embedded in each of its nonzero Z-small sub−module, then $R-$ module C is called Z-small comp, equivalently, C is Z-small comp if there is a monomorphism $f: C \longrightarrow N$ whenever $0 \neq N \ll_z C$). For more on this topic[12-16].

Introducing a Z-small prime module and Z-small prime sub−module where C is called Z-small prime if $ann\ C = ann\ N$, for each nonzero Z-small sub−mod N of C in [8]. And a sub−module N

of an R −module is Z-small prime if whenever $r \in R$, $x \in C$ with $(x) \ll_Z C$ and $rx \in N$ implies either $x \in N$ or $r \in [N{:}C]$. For more on this topic[17-19].

As an extension of Z-small uniform, Z −essentail small uniform ($Z - ess - s$ −uniform module for short) was developed in [8]. An R −module C is called Z-small uniform if every nonzero Z-small sub-module of C is essential. And a R-module C is called Z −essentail small uniform ($Z - ess - s$ −uniform) if for any non zero Z −essentail small sub−module of C is essential, for more information[20-23].

In this paper, as a generalization of the Z-small comp modules, we introduce the basic concept of the $Z - ess - s$ −com modules and examine some of its properties and applications. We will also explore the relationship between the advantages of the $Z - ess - s$ −comp, the Z-small comp modules, the small comp modules, and the comp modules, providing examples and characteristics. We also provide the relationships between $Z - ess - s$ −comp modules, $Z - ess - s$ −prime, and $Z - ess - s$ −uniform.

2. Preliminaries

2.1. Definition:

- Let N be a sub−module of C. If $N + B = C$ with $B \supseteq Z_2(C)$ and $B \leq C$, then $B = C$. This means that N is a Z-small sub−module in C (briefly $N \ll_z C$)[8].
- A sub−module N of a module C is called Z −essentail small sub−module (briefly $N \ll_{Z-ess} C$ if $N + K = C$, $K \leq_{Z-ess} C$, implies$K = C$[11].
- If C embedded in each of its nonzero sub−module. That is for each non-zero sub−module N of C ∃ a monomorphism $f : C \longrightarrow N$. then R − module C is called comp [12].
- If C embedded in each of its nonzero Z-small sub−module, then R − module C is called a Z-small comp equivalently, C is Z-small comp if ∃ a monomorphism $f : C \longrightarrow N$ whenever $0 \neq N \ll_z C$[8].

2.2. Remark

- In generally, every small sub−module is Z-essential small, but the converse is not true. For instance, in the Z-module Z_{24}. Only $\langle\bar{2}\rangle, \langle\bar{4}\rangle, Z_{24}$ are Z −essentail in Z_{24}. Note that $\langle\bar{2}\rangle \ll_{Z-ess} Z_{24}$ since $\langle\bar{2}\rangle + \langle\bar{2}\rangle \neq Z_{24}$, we have $\langle\bar{2}\rangle \ll_{Z-ess} Z_{24}$. On the other hand $\langle\bar{2}\rangle$ is not small in Z_{24}.
- Every Z −essentail small sub−module is an essential small sub−module.
- Every Z −essentail small sub−module is a Z −small sub−module.
- Every small sub−module is a Z −small essential sub−module[11].

2.3. Lemma

Let C be an R −module. The following assertions hold[11]:

- let $N \leq C$ if $N \ll_{Z-ess} C$, then $N \ll_Z C$.
- If $K \ll_{Z-ess} N \leq C$, then $K \ll_{Z-ess} C$.
- If $f{:}\, C_1 \longrightarrow C_2$ is an epimorphism and $N \ll_{Z-ess} C_1$, then $f(N) \ll_{Z-ess} C_2$.
- If $K \leq N \leq C$ and $N \ll_{Z-ess} C$, then $K \ll_{Z-ess} C$.

2.4. Lemma

- If $Z_2(C) = C, N \leq C$, then the following assertions are identical[11]:

$N \ll C$, $N \ll_{ess} C$, $N \ll_{Z-ess} C$

- Let C be a Z-uniform module (i.e. every nonzero sub−module of C is Z-essential)[24], $N \leq C$. Then, $N \ll C$ if and only if $N \ll_{Z-ess} C$.

2.5. Lemma

Let C_1 and C_2 be an R −module, $N_1 \leq C_1$, $N_2 \leq C_2$. Then, $N_1 \ll_{Z-ess} C_1$, $N_2 \ll_{Z-ess} C_2$ if and only if $N_1 \oplus N_2 \ll_{Z-ess} C_1 \oplus C_2$.[11]

3. Z −Essintial Small Compressible Modules

In the section, a generalization of a $Z-ess-s-$ compressible module was given and studied.

3.1. Definition:

An R-module C is said to be Z −essential small compressible ($Z-ess-s-$ comp module for short), if C can be embedded in each of its non zero Z −essential small sub−modules. Equivalenty, C is $Z-ess-s-$ comp if there exists a monomorphism $f: C \longrightarrow N$ whenever $N \ll_{Z-ess} C$.

3.2. Example:

- Since every sub−module of the Z-module $Z_{P\infty}$ is Z- essential small, so $Z_{P\infty}$ is not $Z-ess-s-$ compressible since it can not be embedded in any Z- essential small sub−module.
- Z_{12} as Z-module is not Z-small comp, since $(\bar{2}) \ll_Z Z_{12}$ but there is no one to one homomorphism between Z_{12} and $(\bar{2})$. Also Z_{12} as Z-module is not $Z-ess-s-$comp since $(\bar{2}) \ll_{Zes} Z_{12}$ also there is no one−to−one homomorphism between them.
- Z as Z-module is $Z-ess-s-$comp (by logic) since it is comp.
- Every simple module is $Z-ess-s-$ comp but not conversely, since Z as a Z-module is $Z-ess-s-$ comp but not conversely.
- Z_4 as Z_2-module is not $Z-ess-s-$ comp.

3.3. Proposition:

A module C is $Z-ess-s-$ comp if and only if C can be embedded in Rx for each non zero element x in C and $Rx \ll_{Z-ess} C$.

Proof: ($\Longrightarrow$) by definition (3.1)

($\Longleftarrow$) Let $0 \neq N \leq C$ and let $0 \neq x \in N$, so by assumption $Rx \ll_{Z-ess} C$, and there is a monomorphism $f: C \longrightarrow Rx$. Thus, the composition iof: $C \longrightarrow N$ is a monomorphism where $i: Rx \longrightarrow N$ is the inclusion homomorphism. Therefore, C is a $Z-ess-s-$ comp.

3.4. Remarks:

- Every $Z-ess-s-$ comp is a small comp.

<u>Proof:</u>

Suppose that C is a $Z-ess-s-$ comp and let N be a small sub−module of C. By Remark (2.2), N is Z −essential small and thus by assumption C can be embedded in N.

But the converse is not true: for example, Z_6 as a Z-module is a small comp but not $Z-ess-s-$comp since there is no non zero Z −essential in Z_6 so by logic every sub−module of Z_6 is Z −essential small. But Z_6 cannot be embedded in each of them.

- Every essential small comp is $Z-ess-s-$ comp.

Proof: Suppose that C is an essential small comp and let N be $Z-$ essential small sub−module of C. By Remark (2.2) N is essential small, so by assumption C can be embedded in N. Therefore C is $Z-ess-s-$ comp.

But the converse is not true: for example every sub−module Z_6 as Z_6 is a Z −essential $(\bar{0}), (\bar{2}), (\bar{3})$. To check if they are Z −essential small or not. $(\bar{2}), (\bar{3})$ are not Z −essential small since $(\bar{2}) + (\bar{3}) = Z_6$, $(\bar{2}), (\bar{3}) \leq_{Z-ess} Z_6$ but $(\bar{2}), (\bar{3}) \neq Z_6$, so $(\bar{2}), (\bar{3})$ are not Z −essential small. Then, Z_6 as Z_6 is a $Z-ess-s-$ comp (by logic) since it has no Z −essential small sub−module. But Z_6 as a Z_6- module is not essential small comp since every sub−module of Z_6 is essential small, but Z_6 can not embedded in every non zero essential small.

- Every $Z-$ small comp R −module is $Z-ess-s-$ comp.

Proof: Since every Z −essential small sub−module is Z-small [6] so the statement is true.

The converse of a previous is not true in general: since if C is $Z-ess-s-$ comp. If $N \ll_Z C$ then may be not Z −essential small.

- Every comp module is $Z-ess-s-$ comp.

Proof: clear.

The converse of the above is not true in general since Z_6 as Z_6 is $Z - ess - s -$ comp (by logic) but it is not comp.
We can summarize the above relation by the following **Figure 1**:

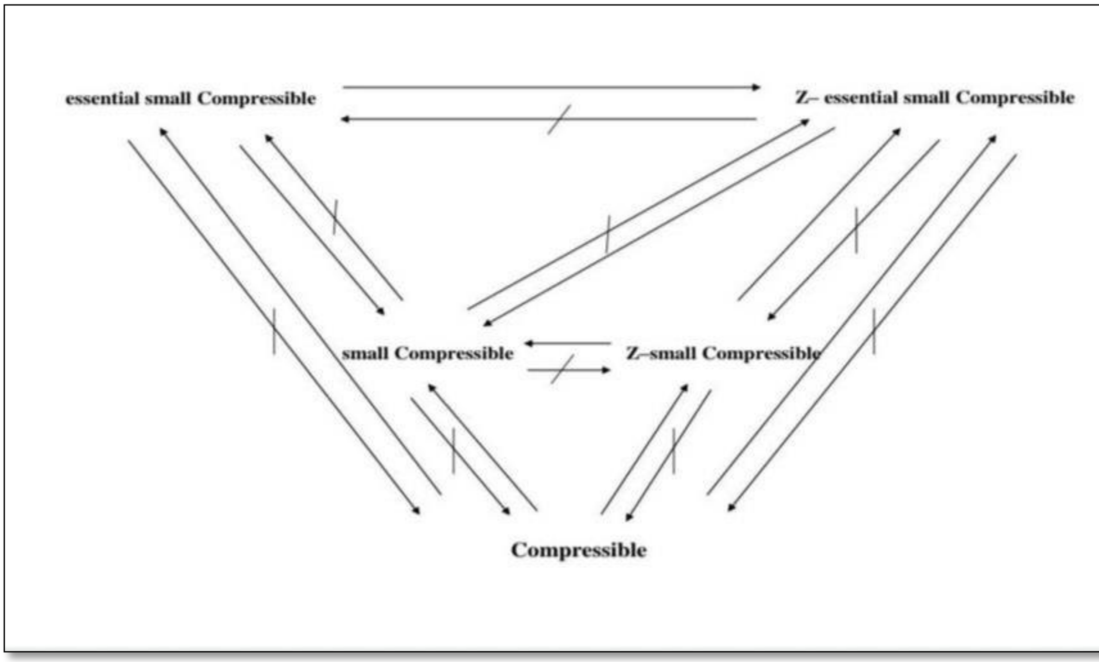

Figure 1: relations between $Z - ess - s -$ comp and other kinds of modules

In the following proposition, we show that the $Z - ess - s -$comp and comp modules are equivalent under certain conditions

3.5. Propositions:

- Let C be a $Z - ess - s -$ comp module, such that every submodule of C contains a non zero $Z -$essential small sub$-$module of C, then C is comp.

Proof: Let $0 \neq N \leq C,$ by assumption N contains a nonzero sub$-$module K such that $K \ll_{Z-ess} N$. By lemma (2.3) we have $K \ll_{Z-ess} C$. But C is $Z - ess - s -$ comp, so there exists a monomorphism $f: C \longrightarrow K$ and, hence $i \circ f: C \longrightarrow N$ is a monomorphism. Therefore, C is comp.

- A $Z - ess - s -$comp module is comp if every cyclic sub$-$module of C is $Z -$essential small in C.

Proof: Let $0 \neq N \leq C$ and let $0 \neq a \in N$, so by assumption $Ra \ll_{Z-ess} C$ and so there is a monomorphism $f: C \longrightarrow Ra$ and hence $i \circ f$: $C \longrightarrow N$ is a monomorphism. Therefore C is a comp.

3.6. Proposition:

A Z- essential small sub$-$module of a $Z - ess - s -$comp module C is also $Z - ess - s -$ comp.

Proof: Let $0 \neq H \ll_{Z-ess} C$ and let $0 \neq K \ll_{Z-ess} N$. Then by lemma (2.3), $K \ll_{Z-ess} C$. But C is $Z - ess - s -$comp, so there exists a monomorphism $L: C \longrightarrow K$ and hence $L \circ i: H \longrightarrow K$ is a monomorphism where $i: H \longrightarrow C$ is the inclusion homomorphism. Hence H is $Z - ess - s -$ comp.

3.7. Remark:

A homomorphic image of a $Z - ess - s -$ comp module need not be a $Z - ess - s -$ comp in general. For example, Z as a Z-module is $Z - ess - s -$ comp but $\frac{Z}{12Z} \simeq Z_{12}$ is not $Z - ess - s -$ comp.

3.8. Proposition:
A direct summand of a $Z-ess-s-$comp module is also $Z-ess-s-$ comp.
Proof: Let $C = L\oplus K$ be a $Z-ess-s-$ comp module and let $0 \neq A \ll_{Z-ess} L$. Thus, by lemma (2.5) then $A\oplus 0 \ll_{Z-ess} C$. By assumption there is a homomorphic f which one−to−one from C to $A\oplus 0$. Its clear $A\oplus 0 \simeq A$, so $f: C \longrightarrow A$ is mono−morphism. Thus, $L \xrightarrow{j_L} C \xrightarrow{f} A$ is a monomorphism where j_L is the injection of L in to C. Hence, L is a $Z-ess-s-$ comp.
3.9. Corollary:
Let C be a semisimple R-module. If C is $Z-ess-s-$ comp, then every nonzero sub−module of C is $Z-ess-s-$ comp.
Proof: Suppose that C is $Z-ess-s-$ comp. Let $0 \neq N \leq C$, so by assumption $N \leq^{\oplus} C$. Thus, by proposition (3.8), so N is $Z-ess-s-$ comp.
3.10. Proposition:
If $C_1 \simeq C_2$ ($C_1$and C_2 are two R-modules). Then C_1 is a $Z-ess-s-$ comp if and only if C_2 is a $Z-ess-s-$ comp.
Proof: Suppose that C_1 is $Z-ess-s-$ comp. Let $\psi: C_1 \longrightarrow C_2$ be an isomorphism, $\psi^{-1}: C_2 \longrightarrow C_1$ is well-be homomorphism. Let $\mathrm{N} \ll_{\mathrm{Z-ess}} C_2$, then $\psi^{-1}(N) \ll_{\mathrm{Z-ess}} C_1$ by lemma (2.3). Put $\mathrm{L} = \psi^{-1}(N)$, $g: C_1 \longrightarrow \mathrm{L}$ is a a monomorphism and $f = \psi|_L$ then $f: \mathrm{L} \longrightarrow C_2$ is a monomorphism. $f(L) = \psi\psi^{-1}(\mathrm{N}) = N$, hence, $f: \mathrm{L} \longrightarrow N$ is a monomorphism. Now, we have the composition $h = f \circ g \circ \psi^{-1}$: $C_2 \xrightarrow{\psi^{-1}} C_1 \xrightarrow{g} L \xrightarrow{f} N$ is a monomorphism. Thus, C_2 is $Z-ess-s-$ comp.
3.11. Proposition:
Let $C = C_1\oplus C_2$ be an R-module such that $ann\, C_1 \oplus ann\, C_2 = R$. Then, C is $Z-ess-s-$ comp if and only if C_1 and C_2 are $Z-ess-s-$ comp.
Proof: ($\Longrightarrow$) its follow from proposition (3.8).
($\Longleftarrow$) Let $0 \neq N \ll_{\mathrm{Z-ess}} C$. Then by[25], $N = K_1\oplus K_2$, for some $0 \neq K_1 \leq C_1 \leq C$ and $0 \neq K_2 \leq C_2 \leq C$. Since $N \ll_{Z-ess} C$ so by lemma (2.5) we have $K_1 \ll_{\mathrm{Z-ess}} C_1$ and $K_2 \ll_{Z-ess} C_2$. But by assumption C_1 and C_2 are $Z-ess-s-$comp, so there is monomorphism, $f: C_1 \longrightarrow K_1$ and $g: C_2 \longrightarrow K_2$. Define $h: C \longrightarrow N$ by $h(a,b) = \big(f(a), g(a)\big)$, it is clear that h is a monomorphism, and hence C is $Z-ess-s-$ comp.
3.12. Corollary:
Let $\{C_i\}_{i=1}^{n}$ be a finite family of a $Z-ess-s-$comp R-module such that $\sum_{i=1}^{n} ann\, C_i = R$. Then, $\bigoplus_{i=1} C_i$ is also $Z-ess-s-$ comp.
Proof: By induction using previous proposition.
3.13. Proposition:
If $Z_2(C) = C$, then, the following are equivalent:
$*$ Small comp.
$**$ Essintial small comp.
$***$ $Z-ess-s-$ comp.
Proof: $(*) \Longrightarrow (**)$ Let $0 \neq N$ be a sub−module of small comp module C such that $N \ll_{ess} C$. Thus, by assumption $N \ll C$. So C can be embedded in N because C is a small comp. Therefore, C is essential small comp.
$(**) \Longrightarrow (*)$ by [14] Remark (3.3, every essential small comp module is a small comp.
$(**) \Longrightarrow (***)$ by Remark (3.4), every essential small comp module is $Z-ess-s-$ comp.
$(***) \Longrightarrow (**)$ let N be nonzero essential small in a module C thus by assumption $N \ll_{Z-ess} C$ and so there is a monomorphism $f: C \longrightarrow N$. Therefore C is an essential small comp.
$(***) \Longrightarrow (*)$ by remark (3.4), every $Z-ess-s-$comp module is a small comp.
$(*) \Longrightarrow (***)$ Let C be a small comp and let $0 \neq N \ll_{Z-ess} C$ by remarks (2.3)$N \ll C$. But C is small comp so there exist monomorphism, $f: C \longrightarrow N$. Thus C is $Z-ess-s$ comp module.
Now, we have the following:

3.14. Proposition:

If C is a Z-uniform module. Then, C is a small comp if and only if C is $Z-ess-s-$comp.

Proof: Clear from lemma (2.4)

4. Z −essential small uniform modules

In this section, we give $a\ Z$ −essential small uniform and some remarks and examples before giving a new conclusion, we need the following concepts and propositions:

4.1. Definition:

Let C be an R −module, C is called a Z −essential small uniform ($Z-ess-s$ −uniform) if for any nonzero Z −essential small sub−module of C is essential.

4.2. Remarks and Examples:

- It is clear that every uniform module is $Z-ess-s$ −uniform. but not converse, for example Z_6 as Z_6 –module is $Z-ess-s$ −uniform (by logic since the only nonzero sub−modules of Z_6 are $\langle\bar{2}\rangle, \langle\bar{3}\rangle$, but $\langle\bar{2}\rangle, \langle\bar{3}\rangle \ll_{Z-ess} Z_6$) while Z_6 ~~is~~ not uniform as Z_6 –module.
- Every Z −uniform module is $Z-ess-s$ −uniform (where a module C is $a\ Z$ −uniform if every non zero sub−module of C is essential[22]).

Proof: Let $0 \neq N \ll_{Z-ess} C$ so by Remark (2.2), $N \ll_Z C$ and since C is Z −uniform so $N \leq_e C$. Therefore, C is $a\ Z-ess-s$ −uniform.

- Z_4 as Z–module is $a\ Z-ess-s$ −uniform (since it is uniform)

4.3. Definition:

Let C be an R −module and $N < C$. N is called $a\ Z$ −essential small prime sub−module ($Z-ess-s$ −prime) if $rx \in N$ with $\langle x\rangle \ll_{Z-ess} C$ ($r \in R, x \in C$ implies either $x \in N$ or $r \in [N{:}C]$).

4.4. Remarks and Examples:

- Every prime sub−module is $a\ Z-ess-s-$ prime sub−module.

Proof: clear.

- Every $Z-ess-s$ −prime sub−module is a small prime sub−module.

Proof: Let $0 \neq N \leq C$ be $a\ Z-ess-s$ −prime such that $rx \in N$ with $\langle x\rangle \ll C$ (where $r \in R, x \in C$) by remark (2.2) $\langle x\rangle \ll_{Z-ess} C$, by assumption either $x \in N$ or $r \in [N{:}C]$).

- Every $Z-$ small prime sub−module is $a\ Z-ess-s$ −prime sub−module.

Proof: Let $0 \neq N < C$ be $Z-$ small prime sub−module of an R-module C such that $rx \in N$ with $\langle x\rangle \ll_{Z-ess} C$ (where $r \in R, x \in C$) by remark (2.2) $\langle x\rangle \ll_Z C$, by assumption either $x \in N$ or $r \in [N{:}C]$).

4.5. Definition:

An R −module C is called $a\ Z$ −essential small prime module ($Z-ess-s$ −prime) if $ann\ C = ann\ N$ for every non zero Z −essential small sub−module $N$of C.

4.6. Remarks and Examples:

- Every prime module is $a\ Z-ess-s-$ prime.

Proof: clear.

- Every $Z-ess-s$ −prime module is small prime.

Proof: By remark (2.2), we get result.

The converse is not true for example: Z_6 as $a\ Z$ −module is a small prime but not Z −essential small prime module since $\langle\bar{3}\rangle \ll_{Z-ess} Z_6$ but $ann(Z_6) = 6Z \neq ann\langle\bar{3}\rangle = 2Z$.

- Every $Z-$ small prime module is $a\ Z-ess-s-$ prime module.

Proof: By Remark (2.2), we get result.

- Every semi simple is a Z −essential small prime, but the converse is not true for example: Z as $a\ Z$ −module is Z −essential small prime but not semi simple.

4.7. Proposition:

If C is a $Z-ess-s-$comp R-module, then, C is $Z-ess-s-$ prime and $Z-ess-s$ −uniform.

Proof: Let $0 \neq N$ be $a\ Z-$essential small sub$-$module of C to show that C is $a\ Z-ess-s-$ prime. Let $I = ann_R N$, thus $IN = (0)$. But C is $Z-ess-s-$ comp so there exists a monomorphism $f: C \longrightarrow N$. Notice that, $f(IC) = If(C) \subseteq IN = (0)$; this means that $I \subseteq ann_R C$, thus $ann_R C = ann_R N$ i.e. $C\ Z-ess-s-$ prime.

To show that C is $Z-ess-s-$uniform, let $0 \neq x \in N$, so by lemma (2.3) $Rx \ll_{Z-ess} C$. Since C is $Z-ess-s-$ comp, there exists a monomorphism $f: C \longrightarrow Rx$. So $f(x) = rx \neq 0$, for some $r \in R$, $r \neq 0$. Take $0 \neq a \in C$ and $f(a) = tx$, for some $t \in R$. Since f is monomorphism, $tx \neq 0$. We have $f(ra) = rf(a) = rtx = trx = tf(x) = f(tx)$. As f is monomorphism, we have $ra = tx$. But $tx \neq 0$, thus $ra \neq 0$ and $ra \in N$, this means that $N \leq_e C$ and, therefore, C is $Z-ess-s-$uniform.

For the converse of last proposition, we give the following theorem as a partial answer.

4.8. Theorem:

Let $C = Rm_1 \oplus Rm_2 \oplus \dots \oplus Rm_k$, where $m_1, m_2, \dots, m_k \in C$. If C is a $Z-ess-s-$uniform and $a\ Z-ess-s-$ prime, then C is $a\ Z-ess-s-$ comp module.

Proof: Let $C = Rm_1 \oplus Rm_2 \oplus \dots \oplus Rm_k$, where $m_1, m_2, \dots, m_k \in C$, let $N \ll_{Z-ess} C$, $N \neq 0$. By assumption C is $Z-ess-s-$uniform so $N \leq_e C$. Thus for $m_i \in C$ $(i = 1, 2, \dots, k)$, $\exists\ t_i \in R - \{0\}$ such that $0 \neq t_i m_i \in N$. Put $t = t_1, \dots, t_k$.

Claim $t \neq 0$. Assume $t = 0$, hence $t_1, \dots, t_k m_k = 0$, thus $(t_1, \dots, t_{k-1})(t_k m_k) = 0$. But $\langle t_k m_k \rangle \leq N \ll_{Z-ess} C$, so by lemma (2.3) $\langle t_k m_k \rangle \ll_{Z-ess} C$ and since C is $Z-ess-s-$ prime, we get $t_1, \dots, t_{k-1} \in ann\, C$. Then, $(t_1, \dots, t_{k-1}) m_k = \langle 0 \rangle$, hence $(t_1 t_2, \dots, t_{k-2})(t_{k-1}\, m_{k-1}) = 0$. This implies $t_1 t_2, \dots, t_{k-1} \in ann\, C$. Since $\langle t_{k-1} m_{k-1} \rangle \ll_{Z-ess} C$ and C is $Z-ess-s-$ prime. Thus, $t_1 t_2, \dots, t_{k-2} m_{k-2} = 0$. By repeating this process we get $t_1 m_1 = 0$ hence, which is a contradiction. Therefore, $t = 0$. Now, define $f: C \longrightarrow N$ by $f(m) = tm$, for each $m \in C$. Clearly $tm \in N$, for $each\ m$.

$\ker f = \langle 0 \rangle$ since: assume $f(m) = \langle 0 \rangle$ for some $m \in C$. Since $m = r_1 m_1 + \dots + r_k m_k$ for some $r_i \in R$, $\forall i = 1, \dots, k$. Hence, $tr_i m_i = 0$, for each $i = 1, \dots, k$. But $tr_1 m_1 = 0$, so we get $(t_1 t_2, \dots, t_k)(r_1\, m_1) = 0$ and hence $(t_2 t_3, \dots, t_k r_1)(t_1\, m_1) = \langle 0 \rangle$ since $\langle t_1 m_1 \rangle \ll_{Z-ess} N$ and C is $Z-ess-s-$ prime, we get $t_2, \dots, t_k r_1 \in ann\, C$ and so $(t_2, \dots, t_k r_1)(\, m_2) = 0$. Thus, by a similar argument, we get $t_3, \dots, t_k r_1 \in ann\, M$. Repeating this process, we have $r_1 \in ann\, C$. Therefore, $r_1 m_1 = 0$, similary $r_i m_i = 0$, $i = 2, \dots, k$. Therefore $m = 0$, that is $\ker f = \langle 0 \rangle$ and hence C is $a\ Z-ess-s-$ comp module.

5. Conclusion

In this paper, we succeeded in presenting the concepts of $Z-ess-s-$ comp, and we showed that it is an expansion of the concept of $Z-$small comp module, and we presented some important relations between $Z-ess-s-$ comp modules and other kinds of modules. We also presented the two concepts $Z-ess-s-$prime, and a $Z-ess-s-$uniform module, and provided the relationships between the $Z-ess-s-$comp modules, $Z-ess-s-$prime module, and $Z-ess-s-$uniform module.

Conflict of Interest

There is no conflict of interest

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