Ibn Al-Haitham Journal for Pure and Applied Sciences
Journal homepage: jih.uobaghdad.edu.iq
PISSN: 1609-4042, EISSN: 2521-3407
IHJPAS. 2026; 39(4)

P-Essential Maximal Submodules

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Received:22/December/2025
Accepted:14/April/2026
Published: 20/October/2026
https://doi.org/10.30526/39.4.4357

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Abstract

Let S be a commutative ring with identity and Q be a unitary left S-module. The main goal of this work is to introduce and study the idea of a P-essential maximal submodule, which is a generalization for maximal submodules, as a proper submodule N of an S-Mod Q is called P-essential maximal (shortly Pe-max) if for any P-essential submodule W of a module Q with $V < W \leq_{Pe} Q$ implies that $W = Q$, where a sub−mod V of an S-Mod Q is called P-essential in Q, if $V \cap E = 0$ such that $E \leq_{pu} Q$, implies $E = 0$. An ideal I of a ring S is called P-essential maximal ideal, if for any Pe-ideal J of S such that $I < J \leq_{Pe} S$ implies $J = S$. And we introduce the concept P-essential local, an S-Mod Q is called P-essential local module (for short Pe-local), if it has only one Pe-maximal submodule, which is contains each proper submodule of Q. And we research some relationships between P-essential maximal submodules and some other related ideas. Moreover, we give P-essential local module and P-essential radical module. Some properties and many results were proved as well as the behaviour of that concept with its localization was studied and shown.

Keywords: maximal submodules, P − essential maximal submodules, P-essential local, P-essential radical.

1. Introduction

Let S be a commutative ring with identity and Q be a unitary left S-module for short (S-Mod). A submodule (sub−mod) V of an S-Mod Q is essential ($V \leq_e Q$), if $V \cap Q \neq 0$, for each non-zero sub−mod V of Q, [1-6]. One of the generalizations of essential called a P-essential as follows, a sub−mod V of an S-Mod Q is called P-essential in Q, if $V \cap E = 0$ such that $E \leq_{pu} Q$, implies $E = 0$, [7]. A sub−mod E of an S-Mod Q is called pure ($E \leq_{pu} Q$), if $IQ \cap E = IE$, for each ideal I of a ring S, [8-9]. A proper sub−mod V of an S-Mod Q is called small ($V \ll Q$) if $V + E = Q$, for any sub−mod E of Q implies $E = Q$, [10-22]. Defined P-essential small, where a sub−mod V is called a P-essential small of Q, $V + E = Q$, for each $L \leq_{Pe} Q$ implies $L = Q$. [23] It is well-known that, a proper (sub−mod) V of an S-Mod Q is called maximal, if whenever W is a sub−mod of Q with $V < W \leq Q$, implies that $W = Q$, many authors generalized the maximal sub−mod, [24-26]. In this paper, we study the concept, P-essential maximal sub−mod that is, a sub−mod V is called P-essential maximal sub−mod of Q for short (Pe-max), if for any P-essential sub−mod W of an S-Mod Q with $V < W \leq_{Pe} Q$ implies that $W = Q$. An ideal I of a ring S is called a P-essential maximal ideal, if for any Pe-ideal J of S such that $I < J \leq_{Pe} S$ implies $J = S$. And we study the concept: an S-Mod Q is called a P-essential local module (for short Pe-local), if it has only one Pe-maximal sub−mod, which is contains each proper sub−mod of Q. When a ring S is called a Pe-local ring, if S is a Pe-local S-Mod.

Since every small sub−mod is P-essential small, this led us to study the idea of a P-essential radical module (shortly Pe- radical). That is the sum of all P-essential small sub−mod, which is a generalization of the Jacobson radical, which is a sum of all small sub−mod in the module; [27,28].

2. P-essential Maximal Submodules

In this section, we defined the concept of P-essential maximal sub−mod as a generalization of maximal sub−mod and studied some properties.

2.1. Definition: A proper sub−mod V of an S-Mod Q is called P-essential maximal (shortly Pe-max) if for any P-essential sub−mod W of a module Q with $V < W \leq_{Pe} Q$ implies that $W = Q$. An ideal I of a ring S is called a P-essential maximal ideal, if for any Pe-ideal J of S such that $I < J \leq_{Pe} S$ implies $J = S$.

2.2. Remarks and Examples:

1. It is clear that every maximal sub−mod is Pe-max.
2. In general, the converse of (1) is not true. For example: Let $Q = 2Z \oplus 2Z$ and $V = 4Z \oplus (0)$ of Q not be a maximal sub−mod of, since $4Z \oplus (0) < 2Z \oplus (0) \leq 2Z \oplus 2Z$ and $2Z \oplus (0) \neq 2Z \oplus 2Z$. But V is Pe-max, since the only sub−mod in Q containing V properly is $2Z \oplus (0)$, which is not a P-essential sub−mod of Q.
3. A submodule of Pe-max sub−mod need not be a Pe-max sub−mod, for example: In Z_{12} as Z-module, the sub−mod $2Z_{12}$ is the Pe-max sub−mod of Z_{12} since $2Z_{12} < Z_{12} \leq_{Pe} Z_{12}$ implies $Z_{12} = Z_{12}$, but $4Z_{12} \leq 2Z_{12}$ is not the Pe-max sub−mod, since $4Z_{12} < 2Z_{12} \leq_{Pe} Z_{12}$, but $2Z_{12} \neq Z_{12}$.
4. The submodule $2Z_4$ in Z_4 as Z- module is Pe-max, since $2Z_4$ is the maximal sub−mod.
5. In Z_6 as Z- module, $3Z_6$ is the Pe-max sub−mod in Z_6, since $3Z_6 < Z_6 \leq_{Pe} Z_6$ and $4Z_6$ is Pe-max sub−mod in Z_6, since $4Z_6 < Z_6 \leq_{Pe} Z_6$ and $Z_6 = Z_6$.
6. Every sub−mod of a semisimple module is Pe-max.
7. Let $E \leq Q$, if $\frac{Q}{E}$ is a simple module, then E is Pe-max sub−mod.

Proof: Suppose $\frac{Q}{E}$ is simple; then E is maximal; then by (i) E is a Pe-max sub−mod.

8. Consider $V \leq W \leq Q$, if V is a Pe-max of W and W is a Pe-max of Q, then it is not necessary that V is a Pe-max of Q. For example, in $Q = Z_{12}$ as Z- module, $V = 4Z_{12}$, $K = 2Z_{12}$ so $4Z_{12} < 2Z_{12} \leq_{Pe} 2Z_{12}$ and $2Z_{12} = 2Z_{12}$ implies that $4Z_{12}$ is a Pe-max sub−mod in $2Z_{12}$, and since $2Z_{12}$ is maximal sub−mod in Z_{12}, hence $2Z_{12}$ Pe-max sub−mod in Z_{12}, but $V = 4Z_{12}$ is not a Pe-max sub−mod in $Q = Z_{12}$, since $4Z_{12} < 2Z_{12} \leq_{Pe} Z_{12}$, but $2Z_{12} \neq Z_{12}$.
9. 4Z in Z as Z- module is not Pe-max sub−mod of Z, since $4Z < 2Z \leq_{Pe} Z$, but $2Z \neq Z$.

2.3. Proposition: Let Q be an F-regular module, then Q has Pe-max sub−mod.

Proof: Since every non-zero F-regular module has a maximal sub−mod[29], then by Remarks and Examples (2.2,1), the result is proved.

2.4. Corollary: Every simple S-Mod, has a Pe-max sub−mod.

Proof: Since every simple S-Mod is F-regular, then by proposition (2.3) we obtained the result.

2.5. Propositions: Every multiplication module contains a Pe-max submodule.

Proof: Since every multiplication module is a module has a maximal sub−mod,[30] then by remarks and examples (2.2,1); it has a Pe-max sub−mod.

2.6. Corollary: Every cyclic S-Mod has a Pe-max sub−mod.

Proof: Since every cyclic S-Mod is multiplication, then by proposition (2.5) the result is done.

2.7. Proposition: If V is a Pe-max sub−mod of an S-Mod Q, with $W \leq V$, then $\frac{V}{W}$ is a Pe-max sub−mod in $\frac{Q}{W}$.

Proof: Suppose that $\frac{E}{W} \leq_{Pe} \frac{Q}{W}$ such that $\frac{V}{W} < \frac{E}{W} \leq_{Pe} \frac{Q}{W}$. Since $\frac{E}{W} \leq_{Pe} \frac{Q}{W}$ then by[23] $E \leq_{Pe} Q$, we get $W < E \leq_{Pe} Q$. But V is a Pe-max sub−mod in Q, then $E = Q$ implies $\frac{E}{W} = \frac{Q}{W}$. Therefore, $\frac{V}{W}$ is a Pe-max sub−mod in $\frac{Q}{W}$.

The converse is not true unless $Q$is a pure simple module and $W \leq V$ is a pure submodule of Q.

2.8. Proposition: Let $f: Q_1 \longrightarrow Q_2$ be an isomorphism. If V is Pe-max sub−mod in Q_2; then $f^{-1}(V)$ is Pe-max sub−mod in Q_1.

Proof: Let $E \leq_{Pe} Q_1$such that $f^{-1}(V) < E \leq_{Pe} Q_1$. To show that $E = Q_1$. Since $E \leq_{Pe} Q_1$ and f is an isomorphism, then $f(E) \leq_{Pe} Q_2$ by[23]. Then, $f\left(f^{-1}(V)\right) < f(E) \leq_{Pe} f(Q_1) = Q_2$, so $V < f(E) \leq_{Pe} Q_2$, but V is Pe-max sub−mod in Q_2 implies that $f(E) = Q_2 = f(Q_1)$, then $f^{-1}(f(E)) = f^{-1}(f(Q_1))$, hence $E = Q_1$. Therefore, $f^{-1}(V)$ is a Pe-max sub−mod in Q_1.

The converse is not true because the inverse image of a P-essential submodule in Q_2 need not be a P-essential submodule in Q_1.

2.9. Proposition: Let V, W be sub−mods of an S-Mod Q with $V \leq W \leq Q$. If V is a Pe-max sub−mod of Q, then W is a Pe-max sub−mod of Q.

Proof: Allow $E \leq_{Pe} Q$ such that $W < E \leq_{Pe} Q$. Since $V \leq W$, then $V < E \leq_{Pe} Q$, but V is a Pe-max sub−mod in Q, implying that $E = Q$. Therefore, W is a Pe-max sub−mod in Q.

The converse is not true in general. For example, in Z_{12} as Z- module, the sub−mod $2Z_{12}$ is a Pe-max sub−mod of Z_{12} since $2Z_{12} < Z_{12} \leq_{Pe} Z_{12}$ implies $Z_{12} = Z_{12}$, but $4Z_{12} \leq 2Z_{12}$ is not a Pe-max sub−mod since $4Z_{12} < 2Z_{12} \leq_{Pe} Z_{12}$, but $2Z_{12} \neq Z_{12}$.

2.10. Corollary: Allow V, W to be a sub−mod s of an S-Mod Q. If $V \cap W$ is a Pe-max sub−mod of Q, then both V and W are Pe-max of Q.

Proof: Since $V \cap W \leq W \leq Q$ and $V \cap W \leq V \leq Q$, then by proposition (2.9) V and W are Pe-max of Q.

The remark 2.4 explains the converse of corollary 2.10.

2.11. Remark: The opposite of corollary (2.10) in generally, is untrue. As an instance, the sub−mod s $2Z$ and $3Z$ in Z as Z- module are Pe-max sub−mod s, but $2Z \cap 3Z = 6Z$, which is not a Pe-max of Z, since $6Z < 2Z \leq_{Pe} Z$, but $2Z \neq Z$.

2.12. Corollary: Let V, W be sub−mod s of an S-Mod Q. If V and W are a Pe-max sub−mod s of Q, then $V + W$ is Pe-max of Q.

Proof: Since $V \leq V + W \leq Q$ and $W \leq V + W \leq Q$, then by proposition (2.9) $V + W$ is Pe-max of Q.

The converse is not true. For example, in Z_{12} as Z- module, $4Z_{12} + 2Z_{12} = 2Z_{12}$ is a Pe-max in Z_{12}, but $4Z_{12}$ is not a Pe-max in Z_{12}, Remarks and Examples (2.2, iv).

2.13. Corollary: If V is a Pe-max sub−mod of an S-Mod Q and I is an ideal of S, then $[V:_Q I]$ is a Pe-max of Q.

Proof: Since $V < [V:_Q I] \leq$ Q, then by proposition (2.9) $[V:_Q I]$ is a Pe-max sub−mod of Q.

2.14. Remark: The opposite of corollary (2.13), in general, is untrue. As instance, Le $M = Z_{12}$ as Z- module and $N = 6Z_{12}$, thus $2Z_{12} = [N:_M I] = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$ is a Pe-max sub−mod of Z_{12}, since $2Z_{12} < Z_{12} \leq_{Pe} Z_{12}$ implies $Z_{12} = Z_{12}$, but $N = 6Z_{12}$ is not a Pe-max sub−mod of Z_{12}, since $6Z_{12} < 2Z_{12} \leq_{Pe} Z_{12}$, but $2Z_{12} \neq Z_{12}$.

2.15. Definition: An S-Mod Q is called a fully Pe-max sub−mod, if every non-zero sub−mod of Q is Pe-maximal. A ring S is said to be fully Pe-max, if every non-zero ideal of S is Pe-max l ideal.

2.16. Remarks and Examples:

1. Every semisimple module is a fully Pe-max sub−mod, hence Z_6 and Z_{10} as Z- module is a fully Pe-max sub−mod.
2. Z_{12} as Z- module is not a fully Pe-max sub−mod. Since $6Z_{12}$ is not a Pe-max sub−mod in Z_{12}. Since $6Z_{12} < 2Z_{12} \leq_{Pe} Z_{12}$ but $2Z_{12} \neq Z_{12}$.

2.17. Lemma: Let Q be a finitely generated faithful multiplication S-Mod. Then, an ideal $I \leq_{Pe} S$ if $IQ \leq_{Pe} Q$.

Proof: $\Leftarrow$) Assume that $I \cap J = (0)$ such that $J \leq_{Pu} S$. To prove that $J = (0)$. $(I \cap J)Q = (0)Q$, then $IQ \cap JQ = (0)$, since $IQ \leq_{Pe} Q$, hence $JQ = (0)$ since Q is finitely generated faithful multiplication, then $J = (0)$. Thus $\leq_{Pe} S$.

$\Rightarrow$) Let $IQ \cap H = (0)$ such that $H \leq_{Pu} Q$. To prove that $H = (0)$. Since Q is multiplication, then $H = JQ$ for some $J \leq S$. Now, we have $IQ \cap JQ = (0)$ then $(I \cap J)Q = (0)Q$ since Q is

finitely generated faithful multiplication, hence by[30, theorem(3.1), P.768] $I \cap J = (0)$. But $I \leq_{Pe} S$ implies that $J = (0)$ then $JQ = (0)$. Thus $H = (0)$ and $IQ \leq_{Pe} Q$.

2.18. Proposition: Let Q be a finitely generated faithful and multiplication S-Mod, then Q is a fully Pe-max sub−mod if and only if S is a fully Pe-max.

Proof: $\Rightarrow$) Let Q be a fully Pe-max sub−mod, and let I be a proper ideal in S and $J \leq_{Pe} S$ such that $I < J \leq_{Pe} S$, hence by lemma (2.17) $IQ < JQ \leq_{Pe} SQ$. Since Q is a fully Pe-max sub−mod, then IQ is Pe-max, which implies $JQ = SQ$. But Q is finitely generated faithful and multiplication, then $J = S$. Therefore, Q is a fully Pe-max.

$\Leftarrow$) Suppose that S is a fully Pe-max. Let $V < Q$ and $W \leq_{Pe} Q$ such that $V < W \leq_{Pe} Q$, since Q is multiplication, hence $V = IQ$, $W = JQ$ and $Q = SQ$, so $IQ < JQ \leq_{Pe} SQ$. Since Q is finitely generated faithful and multiplication, then by lemma (2.17) $I < J \leq_{Pe} S$. But I is a Pe-max in S implies $J = S$, so $JQ = SQ$, thus $W = Q$. Therefore, Q is a fully Pe-max sub−mod.

2.19. Theorem: Let Q be a finitely generated faithful and multiplication S-Mod, and let $V \leq Q$. Then, the following statements are equivalent:

1. V is Pe-max sub−mod of Q.
2. $[V:_S S]$ is a Pe-max ideal of S.
3. $V = IQ$ is Pe-max, for some ideal I of S.

Proof: 1⟶2) Let $J \leq_{Pe} S$ such that $[V:_S S] < J \leq_{Pe} S$. Since Q is a multiplication R-module, hence $V = [V:_S S]Q < JQ \leq_{Pe} SQ = Q$ so $V < JQ \leq_{Pe} Q$. But V is a Pe-max sub−mod of Q implies $JQ = Q$. Thus $[V:_S S]$ is a Pe-max ideal of S.

2⟶3) Let $V \leq Q$, since Q is multiplication, hence $V = [V:_S S]Q$, but $[V:_S S]$ is a Pe-max ideal of S, hence $V = IQ$ is Pe-max, for some ideal I of S.

3⟶1) Let $E \leq_{Pe} Q$ such that $V < E \leq_{Pe} Q$. Since Q is multiplication, then $V = IQ$ and $E = JQ$, for some I, J of S, so $IQ < JQ \leq_{Pe} SQ$. But IQ is a Pe-max, hence $JQ = SQ$, thus $J = S$. Therefore, V is Pe-max sub−mod of Q.

2.20. Theorem: Let Q be an S-Mod such that $Q = Q_1 \oplus Q_2$, where $Q_1$and Q_2 R-modules and $S = ann_S(Q_1) + ann_S(Q_2)$. If $Q_1$and Q_2 are fully Pe-max sub−mod, then Q is a fully Pe-max sub−mod.

Proof: Let $W \leq_{Pe} Q$ and V be a proper sub−mod of Q such that $V < W \leq_{Pe} Q$, since $S = ann_S(Q_1) + ann_S(Q_2)$, then $V = V_1 \oplus V_2$ for some sub−mod V_1 of Q_1 and V_2 of Q_2 also $W = W_1 \oplus W_2$ for some submodule W_1 of Q_1 and W_2 of Q_2, so $V_1 \oplus V_2 < W_1 \oplus W_2 \leq_{Pe} Q_1 \oplus Q_2$ and since $Q_1$and Q_2 are a fully Pe-max sub−mod, then V_1 and V_2 are Pe-max sub−mod of $Q_1$and Q_2 with $W_1 \leq_{Pe} Q_1$ and $W_2 \leq_{Pe} Q_2$,[7] so $V_1 < W_1 \leq_{Pe} Q_1$, $V_2 < W_2 \leq_{Pe} Q_2$ implies that $W_1 = Q_1$ and $W_2 = Q_2$, thus $W_1 + W_2 = Q_1 + Q_2$. Therefore V a is Pe-max sub−mod, hence Q is a fully Pe-max sub−mod. The converse needs not to be true.

Recall that, an S-Mod Q is called local module, if it has a unique maximal sub−mod, [29].

3. P-Essential Local Module

3.1. Definition: An S-Mod Q is called a P-essential local module (for short Pe-local), if it has only one Pe-max sub−mod, which contains each proper sub−mod of Q.

3.2. Examples and Remarks:

1. Z_4 as Z- module is Pe-local since it has only one Pe-max sub−mod.
2. The sub−mod $2Z_{12}$ of Z_{12} as Z- module is Pe-local since it has only one a Pe-max sub−mod, which is $4Z_{12}$ is a Pe-max sub−mod in $2Z_{12}$.
3. Every local R-module has a Pe-max sub−mod by Remarks and Examples (2.2,1).
4. The opposite of (3), in general, is untrue. As instance, Z_6 as Z- module has a Pe-max sub−mod, but it is not a local R-module.
5. Z_6 as Z- module is not Pe-local module since $2Z_6$, $3Z_6$are Pe-max sub−mods of Z_6.

3.3. Proposition: Allow Q be a non-zero multiplication and Pe-local S-Mod. If $V \neq 0$ is Pe-max sub−mods of Q, then V is maximal of Q.

Proof: Since Q is multiplication, hence Q has maximal sub−mod by[30] and Q is Pe-local implies that Q h'as only one Pe-max, which is V. If V is a Pe-max, thus V is a maximal sub−mod in Q.

4. P-Essential Radical Module

4.1. Definition: Let Q be an R-module, the sum of all P-essential small sub−mod of Q is called a P-essential radical module (shortly Pe- radical) and denoted by $Rad_{Pe}(Q)$.
That is $Rad_{Pe}(Q) = \sum\{V: V \ll_{Pe} Q\}$.

4.2. Remarks and Theorems:

1. $Rad_{Pe}(Z_4) = 2Z_4$.
2. $Rad_{Pe}(Z_6) = 2Z_6 + 3Z_6 = Z_6$ since the Pe-small sub−mod is $2Z_6$ and $3Z_6$.
3. $Rad_{Pe}(Z_{10}) = 2Z_{10} + 5Z_{10} = Z_{10}$ since $2Z_{10}$ and $5Z_{10}$ are Pe-small in Z_{10}.
4. $Rad_{Pe}(Z_{12}) = 2Z_{12}$.
5. $Rad_{Pe}(Rad_{Pe}(Q)) \neq Q$.
   For example: $Rad_{Pe}\big(Rad_{Pe}(Z_{12})\big) = Rad_{Pe}(2Z_{12}) \neq Z_{12}$.
6. $Rad_{Pe}(Z) = (0)$ since (0) is the only Pe-small in Z.
7. If Q is semisimple module, then $Rad_{Pe}(Q) = Q$.[2]

4.3. Proposition:
Let Q be an S-Mod. Then, $Rad(Q) \leq Rad_{Pe}(Q)$.
Proof: Clear.

4.4. Remark: The opposite of proposition (4.1) in general, is untrue. As an instance, consider $Rad_{Pe}(Z_6) = Z_6$ and $Rad(Z_6) = (\bar{0})$, clearly $Rad(Z_6) \leq Rad_{Pe}(Z_6)$.

4.5. Theorem: Let $f: Q \longrightarrow Q'$ be an isomorphism, where Q and Q' are S-Mods, then $f(Rad_{Pe}(Q)) \leq Rad_{Pe}(Q')$.
Proof: Suppose that $V \leq Rad_{Pe}(Q)$ and $V \ll_{Pe} Q$, then by[23] $f(V) \ll_{Pe} Q'$. Hence $f(V) \leq Rad_{Pe}(Q')$ for all $V \ll_{Pe} Q$. Thus, $f\big(Rad_{Pe}(Q)\big) = f\sum_{V \ll_{Pe} Q} V = \sum f(V) \leq Rad_{Pe}(Q')$, so $f(Rad_{Pe}(Q)) \leq Rad_{Pe}(Q')$.
The converse is not true because the inverse image of P-essential submodule need not be P-essential.

4.6. Proposition:
Let V be a sub−mod of an S-Mod Q, then $Rad_{Pe}(V) \leq Rad_{Pe}(Q)$.
Proof: Let $W \leq Rad_{Pe}(V)$ such that $W \ll_{Pe} V$, then by[23] $W \ll_{Pe} Q$, Hence, $W \leq Rad_{Pe}(Q)$. Thus, $Rad_{Pe}(V) \leq Rad_{Pe}(Q)$.
The converse is not true in general. For example, Z_6 as $a\ Z$- module, $Rad_{Pe}(Z_6) = Z_6$ and $Rad_{Pe}(2Z_6) = (\bar{0})$. Hence, $Rad_{Pe}(Z_6) \nleq Rad_{Pe}(2Z_6)$.

4.7. Corollary: Let V be a sub−mod of an S-Mod Q, then $Rad_{Pe}(V) \leq Rad_{Pe}(Q) \cap V$.
Proof: Since $Rad_{Pe}(V) \leq V$ and by proposition (4.4), $Rad_{Pe}(V) \leq Rad_{Pe}(Q)$, thus $Rad_{Pe}(V) \leq Rad_{Pe}(Q) \cap V$ and $Rad_{Pe}(Q) \cap V \leq Rad_{Pe}(V)$. Therefore, $Rad_{Pe}(V) = Rad_{Pe}(Q) \cap V$.
The converse is not true. For example, let $Q = Z_4$ as $a\ Z$- module and $V = 2Z_4$. We have $Rad_{Pe}(Z_4) = 2Z_4$ and $Rad_{Pe}(2Z_4) = (\bar{0})$, then $Rad_{Pe}(Z_4) \cap 2Z_4 = 2Z_4$ but $Rad_{Pe}(Z_4) \cap 2Z_4 \nleq 2Z_4$.

4.8. Proposition: Let V and W be sub-mods of an S-Mod Q, then $Rad_{Pe}(V) + Rad_{Pe}(W) \leq Rad_{Pe}(V + W)$.
Proof: Since $V \leq V + W$ and by proposition (4.4), $Rad_{Pe}(V) \leq Rad_{Pe}(V + W)$. By a similar way, we get $Rad_{Pe}(W) \leq Rad_{Pe}(V + W)$. Therefore, $Rad_{Pe}(V) + Rad_{Pe}(W) \leq Rad_{Pe}(V + W)$.

The converse is not true in general. For example, in Z_{12} as Z- module, we know that $Rad_{Pe}(2Z_{12}) = 2Z_{12}$, $Rad_{Pe}(4Z_{12}) = (\bar{0})$ and $Rad_{Pe}(6Z_{12}) = (\bar{0})$, then $Rad_{Pe}(4Z_{12} + 6Z_{12}) = Rad_{Pe}(2Z_{12}) = 2Z_{12}$ and $Rad_{Pe}(4Z_{12}) + Rad_{Pe}(6Z_{12}) = (\bar{0})$. Hence, $Rad_{Pe}(4Z_{12} + 6Z_{12}) \nleq Rad_{Pe}(4Z_{12}) + Rad_{Pe}(6Z_{12})$.

5. Conclusion

In this work, we achieve the introduction of the idea of a pe-maximal sub mod, which is a generalization of the idea of a maximal sub mod. And we presented some important relations between Pe-maximal sub mod and other kinds of modules. We also introduce the two concepts, P-essential local module and P-essential radical module, and provide the relationships between P-essential small sub mod, Pe-maximal sub mod, and P-essential radical module.

Acknowledgment

We gratefully acknowledge the anonymous referees for their perceptive remarks and constructive critiques, which have substantially improved the presentation and quality of this work

Conflict of Interest

None

Funding

None

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