



P-Essential Maximal Submodules

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Abstract

Let S be a commutative ring with identity and Q be a unitary left S -module. The main goal of this work is to introduce and study the idea of a P-essential maximal submodule, which is a generalization for maximal submodules, as a proper submodule N of an S -Mod Q is called P-essential maximal (shortly Pe-max) if for any P-essential submodule W of a module Q with $V < W \leq_{Pe} Q$ implies that $W = Q$, where a sub-mod V of an S -Mod Q is called P-essential in Q , if $V \cap E = 0$ such that $E \leq_{pu} Q$, implies $E = 0$. An ideal I of a ring S is called P-essential maximal ideal, if for any Pe-ideal J of S such that $I < J \leq_{Pe} S$ implies $J = S$. And we introduce the concept P-essential local, an S -Mod Q is called P-essential local module (for short Pe-local), if it has only one Pe-maximal submodule, which is contains each proper submodule of Q . And we research some relationships between P-essential maximal submodules and some other related ideas. Moreover, we give P-essential local module and P-essential radical module. Some properties and many results were proved as well as the behaviour of that concept with its localization was studied and shown.

Keywords: maximal submodules, P – essential maximal submodules, P-essential local, P-essential radical.

1. Introduction

Let S be a commutative ring with identity and Q be a unitary left S -module for short (S -Mod). A submodule (sub-mod) V of an S -Mod Q is essential ($V \leq_e Q$), if $V \cap Q \neq 0$, for each non-zero sub-mod V of Q ,¹⁻⁶. One of the generalizations of essential called a P-essential as follows, a sub-mod V of an S -Mod Q is called P-essential in Q , if $V \cap E = 0$ such that $E \leq_{pu} Q$, implies $E = 0$,⁷. A sub-mod E of an S -Mod Q is called pure ($E \leq_{pu} Q$), if $IQ \cap E = IE$, for each ideal I of a ring S ,⁸⁻⁹. A proper sub-mod V of an S -Mod Q is called small ($V \ll Q$) if $V + E = Q$, for any sub-mod E of Q implies $E = Q$,¹⁰⁻²². Defined P-essential small, where a sub-mod V is called a P-essential small of Q , $V + E = Q$, for each $L \leq_{Pe} Q$ implies $L = Q$.²³ It is well-known that, a proper (sub-mod) V of an S -Mod Q is called maximal, if whenever W is a sub-mod of Q with $V < W \leq Q$, implies that $W = Q$, many authors generalized the maximal sub-mod,²⁴⁻²⁶. In this paper, we study the concept, P-essential maximal sub-mod that is, a sub-mod V is called P-essential maximal sub-mod of Q for short (Pe-max), if for any P-essential sub-mod W of an S -Mod Q with $V < W \leq_{Pe} Q$ implies that $W = Q$. An ideal I of a ring S is called a P-essential maximal ideal, if for any Pe-ideal J of S such that $I < J \leq_{Pe} S$ implies $J = S$. And we study the concept: an S -Mod Q is called a P-essential local module (for short Pe-local), if it has only one Pe-maximal sub-mod, which is contains each proper sub-mod of Q . When a ring S is called a Pe-local ring, if S is a Pe-local S -Mod.

Since every small sub-mod is P-essential small, this led us to study the idea of a P-essential radical module (shortly Pe- radical). That is the sum of all P-essential small sub-mod, which is a generalization of the Jacobson radical, which is a sum of all small sub-mod in the module;^{27,28}.

2. P-essential Maximal Submodules

In this section, we defined the concept of P-essential maximal sub-mod as a generalization of maximal sub-mod and studied some properties.

2.1. Definition: A proper sub-mod V of an S -Mod Q is called P-essential maximal (shortly Pe-max) if for any P-essential sub-mod W of a module Q with $V < W \leq_{Pe} Q$ implies that $W = Q$. An ideal I of a ring S is called a P-essential maximal ideal, if for any Pe-ideal J of S such that $I < J \leq_{Pe} S$ implies $J = S$.

2.2. Remarks and Examples:

1. It is clear that every maximal sub-mod is Pe-max.
2. In general, the converse of (1) is not true. For example: Let $Q = 2Z \oplus 2Z$ and $V = 4Z \oplus (0)$ of Q not be a maximal sub-mod of, since $4Z \oplus (0) < 2Z \oplus (0) \leq 2Z \oplus 2Z$ and $2Z \oplus (0) \neq 2Z \oplus 2Z$. But V is Pe-max, since the only sub-mod in Q containing V properly is $2Z \oplus (0)$, which is not a P-essential sub-mod of Q .
3. A submodule of Pe-max sub-mod need not be a Pe-max sub-mod, for example: In Z_{12} as Z -module, the sub-mod $2Z_{12}$ is the Pe-max sub-mod of Z_{12} since $2Z_{12} < Z_{12} \leq_{Pe} Z_{12}$ implies $Z_{12} = Z_{12}$, but $4Z_{12} \leq 2Z_{12}$ is not the Pe-max sub-mod, since $4Z_{12} < 2Z_{12} \leq_{Pe} Z_{12}$, but $2Z_{12} \neq Z_{12}$.
4. The submodule $2Z_4$ in Z_4 as Z - module is Pe-max, since $2Z_4$ is the maximal sub-mod.
5. In Z_6 as Z - module, $3Z_6$ is the Pe-max sub-mod in Z_6 , since $3Z_6 < Z_6 \leq_{Pe} Z_6$ and $4Z_6$ is Pe-max sub-mod in Z_6 , since $4Z_6 < Z_6 \leq_{Pe} Z_6$ and $Z_6 = Z_6$.
6. Every sub-mod of a semisimple module is Pe-max.
7. Let $E \leq Q$, if $\frac{Q}{E}$ is a simple module, then E is Pe-max sub-mod.

Proof: Suppose $\frac{Q}{E}$ is simple; then E is maximal; then by (i) E is a Pe-max sub-mod.

8. Consider $V \leq W \leq Q$, if V is a Pe-max of W and W is a Pe-max of Q , then it is not necessary that V is a Pe-max of Q . For example, in $Q = Z_{12}$ as Z - module, $V = 4Z_{12}$, $K = 2Z_{12}$ so $4Z_{12} < 2Z_{12} \leq_{Pe} 2Z_{12}$ and $2Z_{12} = 2Z_{12}$ implies that $4Z_{12}$ is a Pe-max sub-mod in $2Z_{12}$, and since $2Z_{12}$ is maximal sub-mod in Z_{12} , hence $2Z_{12}$ Pe-max sub-mod in Z_{12} , but $V = 4Z_{12}$ is not a Pe-max sub-mod in $Q = Z_{12}$, since $4Z_{12} < 2Z_{12} \leq_{Pe} Z_{12}$, but $2Z_{12} \neq Z_{12}$.
9. $4Z$ in Z as Z - module is not Pe-max sub-mod of Z , since $4Z < 2Z \leq_{Pe} Z$, but $2Z \neq Z$.

2.3. Proposition: Let Q be an F-regular module, then Q has Pe-max sub-mod.

Proof: Since every non-zero F-regular module has a maximal sub-mod²⁹, then by Remarks and Examples (2.2,1), the result is proved.

2.4. Corollary: Every simple S -Mod, has a Pe-max sub-mod.

Proof: Since every simple S -Mod is F-regular, then by proposition (2.3) we obtained the result.

2.5. Propositions: Every multiplication module contains a Pe-max submodule.

Proof: Since every multiplication module is a module has a maximal sub-mod,³⁰ then by remarks and examples (2.2,1); it has a Pe-max sub-mod.

2.6. Corollary: Every cyclic S -Mod has a Pe-max sub-mod.

Proof: Since every cyclic S -Mod is multiplication, then by proposition (2.5) the result is done.

2.7. Proposition: If V is a Pe-max sub-mod of an S -Mod Q , with $W \leq V$, then $\frac{V}{W}$ is a Pe-max sub-mod in $\frac{Q}{W}$.

Proof: Suppose that $\frac{E}{W} \leq_{Pe} \frac{Q}{W}$ such that $\frac{V}{W} < \frac{E}{W} \leq_{Pe} \frac{Q}{W}$. Since $\frac{E}{W} \leq_{Pe} \frac{Q}{W}$ then by²³ $E \leq_{Pe} Q$, we get $W < E \leq_{Pe} Q$. But V is a Pe-max sub-mod in Q , then $E = Q$ implies $\frac{E}{W} = \frac{Q}{W}$. Therefore, $\frac{V}{W}$ is a Pe-max sub-mod in $\frac{Q}{W}$.

The converse is not true unless Q is a pure simple module and $W \leq V$ is a pure submodule of Q .

2.8. Proposition: Let $f: Q_1 \rightarrow Q_2$ be an isomorphism. If V is Pe-max sub-mod in Q_2 ; then $f^{-1}(V)$ is Pe-max sub-mod in Q_1 .

Proof: Let $E \leq_{pe} Q_1$ such that $f^{-1}(V) < E \leq_{pe} Q_1$. To show that $E = Q_1$. Since $E \leq_{pe} Q_1$ and f is an isomorphism, then $f(E) \leq_{pe} Q_2$ by²³. Then, $f(f^{-1}(V)) < f(E) \leq_{pe} f(Q_1) = Q_2$, so $V < f(E) \leq_{pe} Q_2$, but V is Pe-max sub-mod in Q_2 implies that $f(E) = Q_2 = f(Q_1)$, then $f^{-1}(f(E)) = f^{-1}(f(Q_1))$, hence $E = Q_1$. Therefore, $f^{-1}(V)$ is a Pe-max sub-mod in Q_1 .

The converse is not true because the inverse image of a P-essential submodule in Q_2 need not be a P-essential submodule in Q_1 .

2.9. Proposition: Let V, W be sub-mods of an S -Mod Q with $V \leq W \leq Q$. If V is a Pe-max sub-mod of Q , then W is a Pe-max sub-mod of Q .

Proof: Allow $E \leq_{pe} Q$ such that $W < E \leq_{pe} Q$. Since $V \leq W$, then $V < E \leq_{pe} Q$, but V is a Pe-max sub-mod in Q , implying that $E = Q$. Therefore, W is a Pe-max sub-mod in Q .

The converse is not true in general. For example, in Z_{12} as Z -module, the sub-mod $2Z_{12}$ is a Pe-max sub-mod of Z_{12} since $2Z_{12} < Z_{12} \leq_{pe} Z_{12}$ implies $Z_{12} = Z_{12}$, but $4Z_{12} \leq 2Z_{12}$ is not a Pe-max sub-mod since $4Z_{12} < 2Z_{12} \leq_{pe} Z_{12}$, but $2Z_{12} \neq Z_{12}$.

2.10. Corollary: Allow V, W to be a sub-mods of an S -Mod Q . If $V \cap W$ is a Pe-max sub-mod of Q , then both V and W are Pe-max of Q .

Proof: Since $V \cap W \leq W \leq Q$ and $V \cap W \leq V \leq Q$, then by proposition (2.9) V and W are Pe-max of Q .

The remark 2.4 explains the converse of corollary 2.10.

2.11. Remark: The opposite of corollary (2.10) in generally, is untrue. As an instance, the sub-mods $2Z$ and $3Z$ in Z as Z -module are Pe-max sub-mods, but $2Z \cap 3Z = 6Z$, which is not a Pe-max of Z , since $6Z < 2Z \leq_{pe} Z$, but $2Z \neq Z$.

2.12. Corollary: Let V, W be sub-mods of an S -Mod Q . If V and W are a Pe-max sub-mods of Q , then $V + W$ is Pe-max of Q .

Proof: Since $V \leq V + W \leq Q$ and $W \leq V + W \leq Q$, then by proposition (2.9) $V + W$ is Pe-max of Q .

The converse is not true. For example, in Z_{12} as Z -module, $4Z_{12} + 2Z_{12} = 2Z_{12}$ is a Pe-max in Z_{12} , but $4Z_{12}$ is not a Pe-max in Z_{12} , Remarks and Examples (2.2, iv).

2.13. Corollary: If V is a Pe-max sub-mod of an S -Mod Q and I is an ideal of S , then $[V:{}_Q I]$ is a Pe-max of Q .

Proof: Since $V < [V:{}_Q I] \leq Q$, then by proposition (2.9) $[V:{}_Q I]$ is a Pe-max sub-mod of Q .

2.14. Remark: The opposite of corollary (2.13), in general, is untrue. As instance, Let $M = Z_{12}$ as Z -module and $N = 6Z_{12}$, thus $2Z_{12} = [N:{}_M I] = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10}\}$ is a Pe-max sub-mod of Z_{12} , since $2Z_{12} < Z_{12} \leq_{pe} Z_{12}$ implies $Z_{12} = Z_{12}$, but $N = 6Z_{12}$ is not a Pe-max sub-mod of Z_{12} , since $6Z_{12} < 2Z_{12} \leq_{pe} Z_{12}$, but $2Z_{12} \neq Z_{12}$.

2.15. Definition: An S -Mod Q is called a fully Pe-max sub-mod, if every non-zero sub-mod of Q is Pe-maximal. A ring S is said to be fully Pe-max, if every non-zero ideal of S is Pe-max ideal.

2.16. Remarks and Examples:

1. Every semisimple module is a fully Pe-max sub-mod, hence Z_6 and Z_{10} as Z -module is a fully Pe-max sub-mod.
2. Z_{12} as Z -module is not a fully Pe-max sub-mod. Since $6Z_{12}$ is not a Pe-max sub-mod in Z_{12} . Since $6Z_{12} < 2Z_{12} \leq_{pe} Z_{12}$ but $2Z_{12} \neq Z_{12}$.

2.17. Lemma: Let Q be a finitely generated faithful multiplication S -Mod. Then, an ideal $I \leq_{pe} S$ if $IQ \leq_{pe} Q$.

Proof: $\Leftarrow$) Assume that $I \cap J = (0)$ such that $J \leq_{pu} S$. To prove that $J = (0)$. $(I \cap J)Q = (0)Q$, then $IQ \cap JQ = (0)$, since $IQ \leq_{pe} Q$, hence $JQ = (0)$ since Q is finitely generated faithful multiplication, then $J = (0)$. Thus $\leq_{pe} S$.

$\Rightarrow$) Let $IQ \cap H = (0)$ such that $H \leq_{pu} Q$. To prove that $H = (0)$. Since Q is multiplication, then $H = JQ$ for some $J \leq S$. Now, we have $IQ \cap JQ = (0)$ then $(I \cap J)Q = (0)Q$ since Q is

finitely generated faithful multiplication, hence by^{30, theorem(3.1), P.768} $I \cap J = (0)$. But $I \leq_{pe} S$ implies that $J = (0)$ then $JQ = (0)$. Thus $H = (0)$ and $IQ \leq_{pe} Q$.

2.18. Proposition: Let Q be a finitely generated faithful and multiplication S -Mod, then Q is a fully Pe-max sub-mod if and only if S is a fully Pe-max.

Proof: $\Rightarrow$) Let Q be a fully Pe-max sub-mod, and let I be a proper ideal in S and $J \leq_{pe} S$ such that $I < J \leq_{pe} S$, hence by lemma (2.17) $IQ < JQ \leq_{pe} SQ$. Since Q is a fully Pe-max sub-mod, then IQ is Pe-max, which implies $JQ = SQ$. But Q is finitely generated faithful and multiplication, then $J = S$. Therefore, Q is a fully Pe-max.

$\Leftarrow$) Suppose that S is a fully Pe-max. Let $V < Q$ and $W \leq_{pe} Q$ such that $V < W \leq_{pe} Q$, since Q is multiplication, hence $V = IQ$, $W = JQ$ and $Q = SQ$, so $IQ < JQ \leq_{pe} SQ$. Since Q is finitely generated faithful and multiplication, then by lemma (2.17) $I < J \leq_{pe} S$. But I is a Pe-max in S implies $J = S$, so $JQ = SQ$, thus $W = Q$. Therefore, Q is a fully Pe-max sub-mod.

2.19. Theorem: Let Q be a finitely generated faithful and multiplication S -Mod, and let $V \leq Q$. Then, the following statements are equivalent:

1. V is Pe-max sub-mod of Q .
2. $[V:_S S]$ is a Pe-max ideal of S .
3. $V = IQ$ is Pe-max, for some ideal I of S .

Proof: $1 \rightarrow 2$) Let $J \leq_{pe} S$ such that $[V:_S S] < J \leq_{pe} S$. Since Q is a multiplication R -module, hence $V = [V:_S S]Q < JQ \leq_{pe} SQ = Q$ so $V < JQ \leq_{pe} Q$. But V is a Pe-max sub-mod of Q implies $JQ = Q$. Thus $[V:_S S]$ is a Pe-max ideal of S .

$2 \rightarrow 3$) Let $V \leq Q$, since Q is multiplication, hence $V = [V:_S S]Q$, but $[V:_S S]$ is a Pe-max ideal of S , hence $V = IQ$ is Pe-max, for some ideal I of S .

$3 \rightarrow 1$) Let $E \leq_{pe} Q$ such that $V < E \leq_{pe} Q$. Since Q is multiplication, then $V = IQ$ and $E = JQ$, for some I, J of S , so $IQ < JQ \leq_{pe} SQ$. But IQ is a Pe-max, hence $JQ = SQ$, thus $J = S$. Therefore, V is Pe-max sub-mod of Q .

2.20. Theorem: Let Q be an S -Mod such that $Q = Q_1 \oplus Q_2$, where Q_1 and Q_2 R -modules and $S = ann_S(Q_1) + ann_S(Q_2)$. If Q_1 and Q_2 are fully Pe-max sub-mod, then Q is a fully Pe-max sub-mod.

Proof: Let $W \leq_{pe} Q$ and V be a proper sub-mod of Q such that $V < W \leq_{pe} Q$, since $S = ann_S(Q_1) + ann_S(Q_2)$, then $V = V_1 \oplus V_2$ for some sub-mod V_1 of Q_1 and V_2 of Q_2 also $W = W_1 \oplus W_2$ for some submodule W_1 of Q_1 and W_2 of Q_2 , so $V_1 \oplus V_2 < W_1 \oplus W_2 \leq_{pe} Q_1 \oplus Q_2$ and since Q_1 and Q_2 are a fully Pe-max sub-mod, then V_1 and V_2 are Pe-max sub-mod of Q_1 and Q_2 with $W_1 \leq_{pe} Q_1$ and $W_2 \leq_{pe} Q_2$,⁷ so $V_1 < W_1 \leq_{pe} Q_1$, $V_2 < W_2 \leq_{pe} Q_2$ implies that $W_1 = Q_1$ and $W_2 = Q_2$, thus $W_1 + W_2 = Q_1 + Q_2$. Therefore V is Pe-max sub-mod, hence Q is a fully Pe-max sub-mod. The converse needs not to be true.

Recall that, an S -Mod Q is called local module, if it has a unique maximal sub-mod,²⁹.

3. P-Essential Local Module

3.1. Definition: An S -Mod Q is called a P-essential local module (for short Pe-local), if it has only one Pe-max sub-mod, which contains each proper sub-mod of Q .

3.2. Examples and Remarks:

1. Z_4 as Z - module is Pe-local since it has only one Pe-max sub-mod.
2. The sub-mod $2Z_{12}$ of Z_{12} as Z - module is Pe-local since it has only one a Pe-max sub-mod, which is $4Z_{12}$ is a Pe-max sub-mod in $2Z_{12}$.
3. Every local R -module has a Pe-max sub-mod by Remarks and Examples (2.2,1).
4. The opposite of (3), in general, is untrue. As instance, Z_6 as Z - module has a Pe-max sub-mod, but it is not a local R -module.
5. Z_6 as Z - module is not Pe-local module since $2Z_6, 3Z_6$ are Pe-max sub-mods of Z_6 .

3.3. Proposition: Allow Q be a non-zero multiplication and Pe-local S -Mod. If $V \neq 0$ is Pe-max sub-mods of Q , then V is maximal of Q .

Proof: Since Q is multiplication, hence Q has maximal sub-mod by³⁰ and Q is Pe-local implies that Q has only one Pe-max, which is V . If V is a Pe-max, thus V is a maximal sub-mod in Q .

4. P-Essential Radical Module

4.1. Definition: Let Q be an R -module, the sum of all P-essential small sub-mod of Q is called a P-essential radical module (shortly Pe- radical) and denoted by $Rad_{Pe}(Q)$.

That is $Rad_{Pe}(Q) = \sum\{V: V \ll_{Pe} Q\}$.

4.2. Remarks and Theorems:

1. $Rad_{Pe}(Z_4) = 2Z_4$.
2. $Rad_{Pe}(Z_6) = 2Z_6 + 3Z_6 = Z_6$ since the Pe-small sub-mod is $2Z_6$ and $3Z_6$.
3. $Rad_{Pe}(Z_{10}) = 2Z_{10} + 5Z_{10} = Z_{10}$ since $2Z_{10}$ and $5Z_{10}$ are Pe-small in Z_{10} .
4. $Rad_{Pe}(Z_{12}) = 2Z_{12}$.
5. $Rad_{Pe}(Rad_{Pe}(Q)) \neq Q$.

For example: $Rad_{Pe}(Rad_{Pe}(Z_{12})) = Rad_{Pe}(2Z_{12}) \neq Z_{12}$.

6. $Rad_{Pe}(Z) = (0)$ since (0) is the only Pe-small in Z .

7. If Q is semisimple module, then $Rad_{Pe}(Q) = Q$.²

4.3. Proposition:

Let Q be an S -Mod. Then, $Rad(Q) \leq Rad_{Pe}(Q)$.

Proof: Clear.

4.4. Remark: The opposite of proposition (4.1) in general, is untrue. As an instance, consider $Rad_{Pe}(Z_6) = Z_6$ and $Rad(Z_6) = (0)$, clearly $Rad(Z_6) \leq Rad_{Pe}(Z_6)$.

4.5. Theorem: Let $f: Q \rightarrow Q'$ be an isomorphism, where Q and Q' are S -Mods, then $f(Rad_{Pe}(Q)) \leq Rad_{Pe}(Q')$.

Proof: Suppose that $V \leq Rad_{Pe}(Q)$ and $V \ll_{Pe} Q$, then by²³ $f(V) \ll_{Pe} Q'$. Hence $f(V) \leq Rad_{Pe}(Q')$ for all $V \ll_{Pe} Q$. Thus, $f(Rad_{Pe}(Q)) = f(\sum_{V \ll_{Pe} Q} V) = \sum f(V) \leq Rad_{Pe}(Q')$, so $f(Rad_{Pe}(Q)) \leq Rad_{Pe}(Q')$.

The converse is not true because the inverse image of P-essential submodule need not be P-essential.

4.6. Proposition:

Let V be a sub-mod of an S -Mod Q , then $Rad_{Pe}(V) \leq Rad_{Pe}(Q)$.

Proof: Let $W \leq Rad_{Pe}(V)$ such that $W \ll_{Pe} V$, then by²³ $W \ll_{Pe} Q$, Hence, $W \leq Rad_{Pe}(Q)$. Thus, $Rad_{Pe}(V) \leq Rad_{Pe}(Q)$.

The converse is not true in general. For example, Z_6 as a Z - module, $Rad_{Pe}(Z_6) = Z_6$ and $Rad_{Pe}(2Z_6) = (0)$. Hence, $Rad_{Pe}(Z_6) \not\leq Rad_{Pe}(2Z_6)$.

4.7. Corollary: Let V be a sub-mod of an S -Mod Q , then $Rad_{Pe}(V) \leq Rad_{Pe}(Q) \cap V$.

Proof: Since $Rad_{Pe}(V) \leq V$ and by proposition (4.4), $Rad_{Pe}(V) \leq Rad_{Pe}(Q)$, thus $Rad_{Pe}(V) \leq Rad_{Pe}(Q) \cap V$ and $Rad_{Pe}(Q) \cap V \leq Rad_{Pe}(V)$. Therefore, $Rad_{Pe}(V) = Rad_{Pe}(Q) \cap V$.

The converse is not true. For example, let $Q = Z_4$ as a Z - module and $V = 2Z_4$. We have $Rad_{Pe}(Z_4) = 2Z_4$ and $Rad_{Pe}(2Z_4) = (0)$, then $Rad_{Pe}(Z_4) \cap 2Z_4 = 2Z_4$ but $Rad_{Pe}(Z_4) \not\leq 2Z_4$.

4.8. Proposition: Let V and W be sub-mods of an S -Mod Q , then $Rad_{Pe}(V) + Rad_{Pe}(W) \leq Rad_{Pe}(V + W)$.

Proof: Since $V \leq V + W$ and by proposition (4.4), $Rad_{Pe}(V) \leq Rad_{Pe}(V + W)$. By a similar way, we get $Rad_{Pe}(W) \leq Rad_{Pe}(V + W)$. Therefore, $Rad_{Pe}(V) + Rad_{Pe}(W) \leq Rad_{Pe}(V + W)$.

The converse is not true in general. For example, in Z_{12} as Z - module, we know that $Rad_{Pe}(2Z_{12}) = 2Z_{12}$, $Rad_{Pe}(4Z_{12}) = (0)$ and $Rad_{Pe}(6Z_{12}) = (0)$, then $Rad_{Pe}(4Z_{12} + 6Z_{12}) = Rad_{Pe}(2Z_{12}) = 2Z_{12}$ and $Rad_{Pe}(4Z_{12}) + Rad_{Pe}(6Z_{12}) = (0)$. Hence, $Rad_{Pe}(4Z_{12} + 6Z_{12}) \not\leq Rad_{Pe}(4Z_{12}) + Rad_{Pe}(6Z_{12})$.

5. Conclusion

In this work, we achieve the introduction of the idea of a pe-maximal sub mod, which is a generalization of the idea of a maximal sub mod. And we presented some important relations between Pe-maximal sub mod and other kinds of modules. We also introduce the two concepts, P-essential local module and P-essential radical module, and provide the relationships between P-essential small sub mod, Pe-maximal sub mod, and P-essential radical module.

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Conflict of Interest

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