Determination of the Minimum Uncertainty States for a Rosen-Morse Potential by Using the Canonical Transformation of the Hamiltonian of the System

W.R. Mohsen
Department of Physics, College of Education, Ibn Al-Haitham, University of Baghdad

Abstract

The wave functions of the minimum uncertainty states for a one-dimensional Rosen-Morse potential are obtained via exploiting Nieto's formalism for the construction of minimum uncertainty coherent states for different one-dimensional potentials and by using a canonical transformation of the Hamiltonian of the problem into a new Hamiltonian which is chosen so that it looks like a harmonic oscillator one. Also, the mathematical derivations of the Nieto's procedures are presented.

Introduction

The concept of coherent states (CS) has been successfully used in the last decade in many different contexts of theoretical and experimental physics, in particular quantum optics(1). Schrödinger first discovered the coherent states for the harmonic oscillator potential in 1926 and much work has been done since then on their properties and applications.

Recently, coherent states have been found in special Hamiltonians(2). These coherent states are called minimum uncertainty coherent states. In coherent states the standard deviation of X (coordinate) and P(momentum) are equal and their product is minimum over states. There are also quantum states, though we have minimum uncertainty for the standard deviation of coordinate and momentum, they are not equal any- more; those states are called squeezed states. These states are as important as coherent ones(3). Here exists three definitions of coherent states for the harmonic oscillator system. The first one defines the usual coherent states as

الممسوحة ضوئيا بـ CamScanner

eigenstates of the annihilation operator $(\bar{a})$ for each individual oscillator mode of the electromagnetic field (4,5)

$$\bar{a}\,|\,z\rangle = z\,|\,z\rangle \text{ ....[1]}$$

where $[\bar{a},\overset{+}{a}] = 1$ $((\bar{a})^{+} = \overset{+}{a})$ and (z) is a complex constant with conjugate $\bar{z}$. The resulting unit normalized states $|z\rangle$ are given by:

$$|\,z\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\,|\,n\rangle \text{ ....[2]}$$

where $|n\rangle$ is an element of the Fock space. A second definition of coherent states for oscillators assumes the existence of a unitary "displacement" operator D(z) defined as

$$D(z) = e^{(z a^{+} - \bar{z}\bar{a})} \text{ ....[3]}$$

The coherent states parametrized by (z) are given by the action of D(z) on the ground state $|0\rangle$(1).

$$D(z)\,|\,0\rangle = |\,z\rangle = e^{(z\overset{+}{a} - \bar{z}\bar{a})}\,|\,0\rangle \text{ ....[4]}$$

A third definition is based on the uncertainty relation, with position (x) and momentum (p) given, as usual, by:

$$\left.\begin{aligned} x &= \frac{1}{\sqrt{2}}(\bar{a} + \overset{+}{a}) \\ p &= \frac{1}{\sqrt{2}}(\overset{+}{a} - \bar{a}) \end{aligned}\right\} \text{ ....[5]}$$

The coherent states defined above have the minimum uncertainty value $\Delta x \Delta p = \frac{1}{2}$ and maintain this relation in time. Coherent states have two important properties. First, they are not orthogonal to each other.

Second, they provide a resolution of the identity, i.e., they form an over-complete set states(4). The Minimum Uncertainty Method to obtain coherent states for general potentials has been applied to general Hamiltonian systems, to obtain both generalized coherent states and generalized squeezed states(6). In this method, the classical variables (x) and (p) of the classical Hamiltonian of the problem are transformed into the "natural classical variables", (X_c) and (P_c) which vary as the sin and cos of the classical ($\Omega_c t$). The classical Hamiltonian is therefore of the form $P_c^2 + X_c^2$. These natural classical variables are next changed into natural quantum operators.

$$\left.\begin{aligned} X &= X_c(x) \\ P &= \frac{1}{2}(X_c' p + pX_c') \end{aligned}\right\} \quad ....[6]$$

These quantum operators have a commutation relation and associated uncertainty relation

$$\left.\begin{aligned} [X,P] &= iG \\ (\Delta X)^2(\Delta P)^2 &\geq \frac{1}{4}\langle G\rangle^2 \end{aligned}\right\} \quad ....[7]$$

where (G) is an operator. The states that minimize this uncertainty relation are given by the solution to the eigenvalue equation

$$\hat{Y}\Psi_{ss} = C\Psi_{ss} \quad ....[8]$$

$$\text{or } \left\{X + i\frac{\langle G\rangle}{2(\Delta P)^2}P\right\}\Psi_{ss} = \left\{\langle X\rangle + i\frac{\langle G\rangle}{2(\Delta P)^2}\langle P\rangle\right\}\Psi_{ss} \quad ....[9]$$

where C is a complex number.

The classical transformation (x) and (p) to the variables (X_c) and (P_c) is not a canonical transformation (7), i.e., $P_c = pX_c'$ is not the canonically conjugate momentum(8). The classical canonical transformation from the variables (x,p) to variables (X_c) and $P_{c(can)}$ is defined by the poisson bracket relation(8).

$[X_c, P_{c(canon)}]=1$ ....[10]

if X_c is only a function of x, then

$P_{c(canon)} = p/X_c'$

Therefore, the canonical quantum operator $P_{(canon)}$ will have the form:

$$P_{canon}=\frac{1}{2i}\left(\frac{1}{X'}\frac{d}{dx}+\frac{d}{dx}\frac{1}{X'}\right) \quad ....[11]$$

Giving

$$\left.\begin{aligned}&[X,P_{canon}]=i\\&(\Delta X)^2.(\Delta P_{canon})^2\geq\frac{1}{4}\end{aligned}\right\} \quad ....[12]$$

One can compare these expressions with those of relations [6] and [7].

Mathematical treatment of the formalism:

The symmetric Rosen-Morse (RM) potential is (9):

$V_{(x)} = V_0 \tanh^2 z$, z=ax ....[13]

The one-dimensional classical Hamiltonian for this potential is:

$$E=\frac{P^2}{2m}+V_0\tanh^2 z$$

$$=\frac{m\dot{x}^2}{2}+V_0\tanh^2 z \quad ....[14]$$

Therefore, $\dot{X}=\left\{\frac{2}{m}(E-V_0\tanh^2 z\right\}^{1/2}$ ....[15]

$$X_c=X_c(x)=X_{c(\max)}\sin\Omega_c t \quad ....[16]$$

$$P_c=m\dot{X}_c=m\Omega_c X_{c(\max)}\cos\Omega_c t \quad ......[17]$$

$$\dot{X}_c=\Omega_c X_{c(\max)}(1-\sin^2\Omega_c t)^{1/2}$$

$$=\Omega_c(X^2_{c(\max)}-X^2_{c(\max)}\sin^2\Omega_c t)^{1/2}$$

$$=\Omega_c(X^2_{c(\max)}-X^2_c)^{1/2} \quad ....[18]$$

المسوحة ضوئيا بـ CamScanner

But, we have $\dot{X}_c = \frac{dX_c}{dt} = \frac{dX_c}{dx}\frac{dx}{dt} = X'_c\dot{x}$ ....[19]

Therefore, $$X'_c = \frac{\dot{X}_c}{\dot{x}} = \left(\frac{\Omega_c^2 m}{2}\right)^{1/2}\left\{\frac{(X^2_{c(\max)} - X_c^2}{(E - V_o \tanh^2 z}\right\}^{1/2}$$

Or $$X'_c = \frac{dX_c}{dx} = \left(\frac{\Omega_c^2 m}{2}\right)^{1/2}\left\{\frac{(X^2_{c(\max)} - X_c^2)}{(E - V_o \tanh^2 z)}\right\} \text{.........[20]}$$

$$\frac{dX_c}{\sqrt{(X^2_{c(\max)} - X_c^2)}} = \left(\frac{\Omega_c^2 m}{2}\right)^{1/2}\frac{dx}{\sqrt{(E - V_o \tanh^2 z)}} \text{.......[21]}$$

we have, $\operatorname{sech}^2 z + \tanh^2 z = 1$, therefore equation (21) becomes,

$$\frac{dX_c}{\sqrt{(X^2_{c(\max)} - X_c^2}} = \left(\frac{\Omega_c^2 m}{2}\right)^{1/2}\frac{dx}{\sqrt{\mathrm{E}\operatorname{sech}^2 z + \mathrm{E}\tanh^2 z - \mathrm{V_o}\tanh^2 z}}$$

$$= \left(\frac{\Omega_c^2 m}{2}\right)^{1/2}\frac{dx}{\sqrt{\mathrm{E}\operatorname{sech}^2 z - (\mathrm{V_o} - \mathrm{E})\tanh^2 z}}$$

$$= \left(\frac{\Omega_c^2 m}{2}\right)^{1/2}\frac{dx}{\sqrt{(V_o - E)\operatorname{sech}^2 z\left(\frac{E}{V_o - E} - \frac{\tanh^2 z}{\operatorname{sec}h^2 z}\right)}}$$

$$= \left\{\frac{\Omega_c^2 m}{2a^2(V_o - E)}\right\}^{1/2}\frac{dz}{\operatorname{sec}hz\sqrt{(X^2_{c(\max)} - \sinh^2 z)}} \text{.......[22]}$$

where

$$\left.\begin{aligned} \Omega_c &= \frac{2a^2}{m}(V_o - E) \\ X_{c(\max)} &= \frac{E}{(V_o - E)} \end{aligned}\right\} \text{..........[23]}$$

المسوحة ضوئيا بـ CamScanner

If we let $y^2 = \sinh^2 z$, then equation (22) becomes

$$\frac{dX_c}{\sqrt{(X_{c(\max)}^2 - X_c^2)}} = \frac{ydy}{\sinh z \ \cosh z \ \sec hz\sqrt{(X_{c(\max}^2 - y^2)}}$$

$$= \frac{dy}{\sqrt{(X_{c(\max)}^2 - y^2)}} \quad ....[24]$$

Therefore,

$$\frac{d\left(\frac{X_c}{X_{c(\max)}}\right)}{\sqrt{1-\left(\frac{X_c}{X_{c(\max)}}\right)^2}} = \frac{d\left(\frac{y}{X_{c(\max)}}\right)}{\sqrt{1-\left(\frac{y}{X_{c(\max)}}\right)^2}} \quad ....[25]$$

From which, one get

$$\sinh^{-1}\left(\frac{X_c}{X_{c(\max)}}\right) = \sinh^{-1}\left(\frac{y}{X_{c(\max)}}\right)$$

or $X_c = y = \sinh z$ ....[26]

$P_c = m \dot{X}_c = m\dot{x}a \ \cosh z$

$= ap \cosh z$.........[27]

Also, we have $P_c = m\Omega_c X_{c(\max)} \cos(\Omega_c t)$

$$= m.\sqrt{\frac{2a^2}{m}(V_o - E)}.\sqrt{\frac{E}{V_o E}} \cos(\Omega_c t)$$

$$= \sqrt{2a^2 mE} \cos(\Omega_c t) \quad ....[28]$$

The classical equations of motion are:

$$\dot{X}_c = \dot{x}a \ \cosh z = \frac{m\dot{x}a \ \cosh z}{m}$$

$$= \frac{pa\cosh z}{m}$$

Therefore,

$$\dot{X}_c = \frac{P_c}{m} \quad ....[29]$$

$$\dot{P}_c = \frac{d}{dt}\left\{(2ma^2E)^{1/2}\cos(\Omega_c t)\right\}$$

$$= (2ma^2E)^{1/2}\Omega_c(-\sin(\Omega_c t))$$

$$= -(2ma^2E)^{1/2}.\left(\frac{2a^2}{m}(V_o - E)\right)^{1/2} \sin(\Omega_c t)$$

$$= -2a^2\{(V_o - E)E\}^{1/2}\left\{\frac{(V_o - E)}{(V_o - E)}\right\}^{1/2} \sin(\Omega_c t)$$

$$= -2a^2(V_o - E)\left\{\frac{E}{(V_o - E)}\right\}^{1/2} \sin(\Omega_c t)$$

$$= -2a^2(V_o - E)X_{c(\max)} \sin(\Omega_c t)$$

Therefore,

$$\dot{P}_c = -2a^2(V_o - E)X_c \quad ....[30]$$

It is clearly shown that if (E) is greater than (V_o), the classical particle is unconfined. The equations of motion are still the same, however, the solutions for (X_c) and (P_c) are

$$X_c(E > V_o) = \left\{\frac{E}{(E - V_o)}\right\}^{1/2} \sinh\left\{\frac{2a^2}{m}(E - V_o)\right\}^{1/2} t \quad ....[31]$$

$$P_c(E > V_o) = (2ma^2E)^{1/2}\cosh\left\{\frac{2a^2}{m}(E - V_o)\right\}^{1/2} t \quad ....[32]$$

The quantum operators (X and P) of the classical variables (X_c and P_c) are:

الممسوحة ضوئيا بـ CamScanner

X = $X_c(x)$= sinhz……..[33]

$$p=\frac{1}{2}(X'_c p+pX'_c),\ X'_c=\frac{dX_c}{dx}=a\cosh z$$

$$=\frac{1}{2}\left(a\cosh z(-i\hbar)\frac{d}{dx}+(-i\hbar)\frac{d}{dx}a\cosh z\right)$$

$$=\frac{-i\hbar a^2}{2}\left(\cosh z\frac{d}{dz}+\frac{d}{dz}\cosh z\right)$$

$$=\frac{1}{2i}\left(\cosh z\frac{d}{dz}+\frac{d}{dz}\cosh z\right)\hbar a^2 \quad \text{……..[34]}$$

where $(\hbar a^2)$ is taking out of (P) to make it dimensionless. The commutator of the operators (X and P) is

[X,P] = XP-PX

$$=\frac{1}{2i}\left\{\sinh z\ \cosh z\frac{d\Psi}{dz}+\sinh z\ \cosh z\right.$$

$$\frac{d\Psi}{dz}+\Psi\sinh z\frac{d}{dz}(\cosh z)-\cosh z\ \sinh z\frac{d\Psi}{dz}$$

$$-\Psi\cosh z\frac{d}{dz}(\sinh z)-\Psi\cosh z\frac{d}{dz}(\sinh z)-$$

$$\left.\Psi\sinh z\frac{d}{dz}(\cosh z)-\cosh z\ \sinh z\frac{d\Psi}{dz}\right\}$$

$$=-\frac{1}{2i}.2\cosh z\frac{d}{dz}(\sinh z)$$

= i $\cosh^2$x

Therefore, [X,P]=i $\cosh^2$z ………[35]

Comparing equation [35] with equation [7] one can obtain

G= $\cosh^2$z………[36]

The uncertainty relation is:

المسوحة ضوئيا بـ CamScanner

$$(\Delta X)^2(\Delta P)^2 \geq \frac{1}{4}\langle \cosh^2 z\rangle \quad ......[37]$$

Minimum-uncertainty canonical states

For a canonical transformation from (x and p) to the classical variables (X_c and P_c) we have the definition (8).

$$\left.\begin{aligned} \{X_c, P_c\}_{x,p} &= 1 \\ P_c &= p / X'_c \end{aligned}\right\} \quad ....[38]$$

Therefore.

$$P = \frac{1}{2}(X'_c p + pX'_c)$$

$$= \frac{1}{2}\left(\frac{p}{P_c}.p + p.\frac{p}{P_c}\right)$$

$$= \frac{1}{2}\left\{\frac{m\dot{x}}{m\dot{X}_c}(-i\hbar)\frac{d}{dx} + (-i\hbar)\frac{d}{dX}\frac{m\dot{x}}{m\dot{X}_c}\right\}$$

$$= \frac{-i\hbar}{2}\left\{\frac{dx/dt}{dX_c/dt}\frac{d}{dx} + \frac{d}{dx}\frac{dx/dt}{dX_c/dt}\right\}$$

$$= -\frac{i\hbar}{2}\left\{\frac{1}{dX_c/dx}\frac{d}{dx} + \frac{d}{dx}\frac{1}{dX_c/dx}\right\}, \frac{dX_c}{dx} = X'_c$$

Therefore.

$$P_{(can)} = \frac{1}{2i}\left(\frac{1}{X'_c}\frac{d}{dx} + \frac{d}{dx}\frac{1}{X'_c}\right)\hbar \quad ....[39]$$

For a Rosen- Morse (RM) potential:

$X_c = X = \sinh z$, $X'_c = a \cosh z$

$$P_{(can)} = \frac{1}{2i}\left(\frac{1}{a\cosh z}\frac{d}{dx} + \frac{d}{dx}\frac{1}{a\cosh z}\right)$$

الممسوحة ضوئيا بـ CamScanner

$$=\frac{1}{2i}\left(\frac{1}{\cosh z}\frac{d}{dz}+\frac{d}{dz}\frac{1}{\cosh z}\right)\text{........[40]}$$

Also $P_{(can)}\Psi=\frac{1}{2i}\left(\frac{1}{\cosh z}\frac{d\Psi}{dz}+\frac{d}{dz}\left(\frac{\Psi}{\cosh z}\right)\right)$

$$=\frac{1}{2i}\left(\frac{1}{\cosh z}\frac{d\Psi}{dz}+\frac{1}{\cosh z}\frac{d\Psi}{dz}+\Psi(-1)\cosh_{z}^{-2}\sinh z\right)$$

$$=\frac{1}{2i}\left(\frac{2}{\cosh z}\frac{d\Psi}{dz}-\Psi\frac{\sinh z}{\cosh^2 z}\right)$$

Therefore,

$$P_{(can)}=\frac{1}{i}\left(\frac{1}{\cosh z}\frac{d}{dz}-\frac{\sinh z}{2consh^2 z}\right)\text{..........[41]}$$

$$[X,P_{(can)}]=\frac{1}{i}\left\{\frac{\sinh z}{\cosh z}\frac{d\Psi}{dz}-\frac{\sinh^2 z}{2\cosh^2 z}\Psi\right.$$

$$\left.-\frac{1}{\cosh z}\frac{d}{dz}(\sinh z\Psi)+\frac{\sinh^2 z}{\cosh^2 z}\Psi\right\}$$

$$=\frac{1}{i}\left\{\frac{\sinh z}{\cosh z}\frac{d\Psi}{dz}-\frac{\sinh z}{\cosh z}\frac{d\Psi}{dz}-\frac{\Psi}{\cosh z}\frac{d}{dz}(\sinh z)\right\}$$

$$=-\frac{1}{i}.\frac{1}{\cosh z}.\cosh z=i$$

Therefore, $[X, P_{(can)}]=i$..........[42]

So, we have

$$\left.\begin{array}{l}G=1\\(\Delta X)^2(\Delta P_{(can)})^2\geq\frac{1}{4}\end{array}\right\}\text{.........[43]}$$

The eigenvalue equation for minimum uncertainty states for these canonical operators is (See equation [9]):

$$\left\{X + i\frac{1}{2(\Delta P_{can})^2}P_{(can)}\right\}\Psi_{mus} = C\Psi_{mus} \quad ....[44]$$

where, C=u +iV, is a complex number. If we let, $B=1/2(\Delta P_{can})^2$, then equation (44) becomes,

$$\left\{\sinh z + \frac{B}{\cosh z}\frac{d}{dz} - \frac{B}{2}\frac{\sinh z}{\cosh^2 z}\right\}\Psi_{mus} = C\Psi_{mus}$$

or $$\left\{\sinh z\left(1 - \frac{B}{2\cosh^2 z}\right) + \frac{B}{\cosh z}\frac{d}{dz}\right\}\Psi_{mus} = C\Psi_{mus}$$

so, $$\left\{-\sinh z\left(1 - \frac{B}{2\cosh^2 z}\right) + C\right\}\Psi_{mus} = \frac{B}{\cosh z}\frac{d\Psi_{mus}}{dz}$$

Therefore,

$$\frac{d\Psi_{mus}}{\Psi_{mus}} = \left(-\frac{\sinh z \ \cosh z}{B} + \frac{\sinh z}{2\cosh z} + \frac{C}{B}\cosh z\right)dz$$

$$\int\frac{d\Psi_{mus}}{\Psi_{mus}} = -\frac{1}{B}\int\sinh z \ \cosh z \ dz + \frac{1}{2}\int\tanh z \ dz + \frac{C}{B}\int\cosh z \ dz + K$$

$$Ln\Psi_{mus} = -\frac{\sinh^2 z}{2B} + \frac{C}{B}\sin z + \frac{1}{2}Ln \ \cosh z + K$$

Therefore, Ψ_{mus} =

$$e^{-\sinh^2 z/2B + C\sinh z/B + \frac{1}{2}Ln \ \cosh z + K}$$

$$\Psi_{mus} = N(\cosh^{1/2} z)e^{-\sinh^2 z/2B + C\sinh z/B}$$

الممسوحة ضوئيا بـ CamScanner

where , Ψ_{mus} represents the minimum-uncertainty canonical states for a Rosen-Morse potential and (N) is the normalization factor which contains the integrating constant K.

Discussion

The general formalism of Nieto and Simmons(6,8,9) for the construction of minimum uncertainty coherent states for different one-dimensional potentials, both confining and non-confining in one and three dimensions, is based on the fact that the variables X and P need not be related to the old variables x and p by a canonical transformation. This means that the value of (G) of equation [36] must satisfy the eigenvalue problem of equation [9]. In contrast, the present work makes use of the canonical transformation to relate between the two sets of variables in order to construct the minimum uncertainty states and this means that the value of (G) of equation (43) must satisfy the eigenvalue problem of equation [9]. The comparison between the above two sets of minimum uncertainty states and evaluating normalization factor N, complex number C and the real quantity B of equation [47] needs further investigation and that is a subject of another paper.

References

1. El-Kinani, A.H. and Daoud, M. (2002). Int. J. Mod. Phys., B 16(26): 3915-3937.
2. Nieto, M.M. and Simmons, L.M. (1987). Phys. Rev. Lett., 41: 207.
3. Henry, R.W. and Glotzer, S.C. (1988). Am. J. Phys., 56(2): 163.
4. El-Kinani, A.H. and Daoud, M. (2002). J.Math. Phys., 43(2): 714.
5. Novaes, M.; de Aguiar, M.A.M. and Hornos, J.E.M. (2003). J.Phys., A 36: 5773-5786.
6. Nieto, M.M. (2001). Mod. Phys. Lett., A16(35): 2305-2311.
7. Kope, D.H. (1988). Am. J. Phys., 56(3): 252.
8. Nieto, M.M. and Simmons, L.M. (1979). Phys.Rev. D 20(6): 1321.
9. Nito, M.M. and Simmons, L.M. (1979). Phys. Rev., D20(6): 1342.

CamScanner الممسوحة ضوئيا بـ

تعيين حالات اللادقة الصغرى لجهد روسن-مورس باستخدام التحويلات القانونية لدالة الطاقة

وائل رشيد محسن القسام
قسم الفيزياء، كلية التربية- ابن الهيثم، جامعة بغداد

الخلاصة

تم في البحث الحالي الحصول على حالات اللادقة الصغرى لجهد روسن- مورس في بعد واحد وذلك من خلال استخدام صياغة نيتو وجماعته للحالات المتشاكهة لجهود عامة مختلفة وفي بعد واحد وعن طريق استخدام التحويلات القانونية لدالة الطاقة الكلاسيكية لجهد المسألة الى دالة طاقة جديدة تكافأ تلك التي للمتذبذب التوافقي البسيط. لقد تم ايضاً اجراء الاشتقاقات الرياضية المفصلة لتوضيح هذه الصياغة للجهد المذكور.

الممسوحة ضوئيا بـ CamScanner