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Bifurcation Analysis of Food Chain with Release of Fear, Additional Food and Anti-predator Behavior for the Top Predator Species

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Abstract

This paper investigates the bifurcation analysis of a three-species food chain model with a Holling type II functional response, focusing on the effect of fear and additional food on both middle and top predators, respectively, accompanied by the presence of anti-predatory behavior. The system boundedness had been proven, the existence of all the equilibrium under certain conditions examined, the local stability of the system's equilibrium point demonstrated, and local and Hopf bifurcations have been proved with certain determinations according to specific parameters included in the food chain model. The theoretical results of the system were simulated numerically with phase diagrams to assure local stability at a specific set of the parameters' values with time series for the included populations. Furthermore, the system's bifurcations were verified around the interior equilibrium point during growth parameter versus prey population, creating saddle-node bifurcation, as well as top predator versus predation parameter through Hopf bifurcation. Finally, the conclusion presents the most important results and future recommendations that were submitted.

Keywords: Bifurcation analysis, Food chain, Fear, Additional food, anti-predator.

1. Introduction

A lot of dynamical systems models contain parameters, and a small change in these parameters has a decisive influence on the behavior of the system. In particular, some equilibrium points can disappear, some new ones can be generated, and the stability of some of them could be changed. These arbitrary changes in dynamics are called "bifurcation," and the parameter's value at this occurrence is called a bifurcation point. Bifurcation analysis has a wide application in many scientific fields, for instance, fluid mechanics, chemistry, etc. The most important of these applications is ecology, and food chains are substantial to study ecosystems; they illustrate how different species interact inside an ecosystem. Many considerable works on predator-prey mathematical models were studied [1-3], the importance of studying them aided the development of mathematical models for many ecological systems, leading to the discovery of different dynamical behaviors, including periodic patterns and chaos. Numerous mathematicians and ecologists have expanded upon Lotka's model[4-10], most of the previous research has focused on studying local and Hopf bifurcation that is concerned with the dynamics of ecological systems by examining food chain models with three-dimensional food chain models. The influence of fear, additional food, and anti–predator behavior has been studied widely. The effect of fear on the dynamics of prey, predator, and food chain models is a phenomenon of defensive strategy that inspires the dynamics between predator and prey, ensuring their mutual relationship within the food chain survives and continues. The availability of the prey (middle predator) to the top

predator may decrease due to fear of predation; therefore, both the top predator and middle predator species experience food shortages here comes the role of additional food to ensure the survival of the top predator[11]. Researchers considered a predator–prey model where the prey species faces a reduction in their growth through fear, and the predator has a standby food source that helps the prey to hide in a safer place[12]. While a three-species food chain model assumed that both the birth rate and interspecies competition of prey are impacted by fear generated by the middle predator, and in addition, the food search behavior of the middle predator is forced adequate to the presence of the top predator[13], this model established that the functional responses are stringent for real-world ecological models. The significance of functional responses[14] had been proven theoretically; it presumed the impact on real-world ecological models defined as the predator's intake rate based on prey density. Holling defined three types of functional responses[15]: linear (Holling type I), hyperbolic (Holling type II), and sigmoid (Holling type III)[16], Holling type II functional responses are regularly employed to model predator-prey interactions. A study of a three-species predator-prey system with supplemented food for the top predator found the middle predator acted as prey and a predator, and the top predator harmed both the prey and the middle predator[3], subsequent another work revealed the predator exhibiting refuge behavior towards the top predator and a specific amount of added food supplied to the top predator in a three-species food chain with the Beddington–DeAngalis functional response, utilizing resources partly from the extra food, depending on the population, to simulate the shape of the interaction between three species, prey, predator, and top predator, depending on the fear of the predator in prey species[9]. The anti-predatory behavior is important; many researchers have studied it. They identify two main ways prey exhibit such behaviors: through morphological[17,18] or behavioral changes or through active attacks[19-21].

The main objective of this research is to analyze the bifurcation of a food chain model we combined using two unlike Holling-Type II functional responses. It is found out that fear is a factor for the middle predators from the top predator's influence on the middle predator community, clearly. The disruption of anti-predatory behavior in case of aggravation and providing additional food for the top predator in the system's dynamics requires a secondary food supply for the top predator as a contributing factor to the balance, ensuring the middle and top predator community's survival. All of the above were detailed in the section. 2 on system modeling and formulation, which have submitted the model formulation and proof of its boundedness, with the existence of the possible equilibrium points proven, respectively. In section 3, an examination of the equilibrium points' local stability evidences. The local stability of the equilibrium points introduced in the section. 4, section. 5 examined Hopf-bifurcations for the assumed interior equilibrium point. A numerical simulation and a graphical presentation of the solutions and the dynamic behavior over a time series displayed through section six to support theoretical results and demonstrate the parameters' effect. Lastly, a conclusion about theoretical and numerical findings is given in section.7.

2. Model Formulation, Boundedness, and Equilibrium Point Existence

The mathematical model formulation was adopted in this section, it represents the interaction between three layers of food chain species, *s(t)* is the prey population density at time *t* of the first level in the food chain growth logistically, *p(t)* the population density at time *t* of middle predator (second level of the food chain species) with total Holling- type II total predation functional response of prey by the middle predator with presence of fear β parameter form the top predator where $\beta \geq 0$ and $\frac{1}{1+\beta q}$ symbolize the fear function of the middle predator from the top predator, and *q(t)* adapt the population density at time *t* of top predator (third level of the food chain species) the terms $\frac{\gamma_2 pq}{a_2+\alpha e(1-A)+p}$ and $\frac{\gamma_2(p+e(1-A))q}{a_2+\alpha e(1-A)+p}$ expound the predation Holling –type II functional response of middle predator via top predator in the food chain include the

addition food (1-A**)** for the top predator, ultimately $\frac{\gamma_2(p+e(1-A))q}{a_2+\alpha e(1-A)+p}$ is the total predation of middle predator via the top predator. The following set of ODE simulate the dynamics of the above ecological food chain system

$$\begin{aligned}\frac{ds}{dt} &= rs\left(1-\frac{s}{k}\right)-\frac{\gamma_1 sp}{(a_1+s)(1+\beta q)}\\ \frac{dp}{dt} &= \frac{\gamma_1 sp}{(a_1+s)(1+\beta q)}-\delta_1 p-\frac{\gamma_2 pq}{a_2+\alpha e(1-A)+p}\\ \frac{dq}{dt} &= \frac{\gamma_2(p+e(1-A))q}{a_2+\alpha e(1-A)+p}-\delta_2 q-\mu pq\end{aligned} \quad \textbf{(1)}$$

We can rewrite the system in the form

$$\begin{aligned}\frac{ds}{dt} &= s\left[r\left(1-\frac{s}{k}\right)-\frac{\gamma_1 p}{(a_1+s)(1+\beta q)}\right]=s\varphi_1(s,p,q)\\ \frac{dp}{dt} &= p\left[\frac{\gamma_1 s}{(a_1+s)(1+\beta q)}-\delta_1-\frac{\gamma_2 q}{a_2+\alpha e(1-A)+p}\right]=p\varphi_2(s,p,q)\\ \frac{dq}{dt} &= q\left[\frac{\gamma_2(p+e(1-A))}{a_2+\alpha e(1-A)+p}-\delta_2-\mu p\right]=q\varphi_3(s,p,q)\end{aligned} \quad \textbf{(2)}$$

With $s(t)\geq 0, p(t)\geq 0, q(t)\geq 0$,. In so far we observe that if $A=0$ the total number of the top predator became $\frac{\gamma_2 pq}{a_2+\alpha e+p}$ and drops , in contrast, the availability of additional food $A=1$, the growth rate of the top predator increase to $\frac{\gamma_2 pq}{a_2+p}$. Clearly the role of additional food becomes more effective which constitutes as a substitutional that reduces the need for predation, it is of animated importance for top predator, with fear present of the middle predator to recover the food reduction. The positive constants r is the logistic growth rate of the prey, k is the utmost number of prey species that an environment can support , δ_1, δ_2 describe death rate of the middle and top predator's respectively, a_1, a_2 are the middle and top predator's half saturation respectively, e due the conversion rate, α represent the search rate for the additional food by the top predator, γ_1, γ_2 are the aggression rate of the middle and top predator's respectively considering that $\gamma_2\geq\gamma_1$, μ represent the anti-predator constant taken as a protective behavior.

The next theorem guarantees that solutions of **System 2** remain within a certain range whatever the initial conditions.

2.1. Theorem The solutions of **System 2** are uniformly bounded and enter into the region $\Omega=\left\{(s,p,q)\in\mathbb{R}^3_+;\ \omega(t)\leq\frac{V}{m}\right\}$

Proof. From **System 2** we have $\frac{ds}{dt}\leq rs\left(1-\frac{s}{k}\right)$ for this it is simple to establish that $s(t)\leq 1$ when $t\to\infty$, now let $\omega(t)$ defined as the total population of **System (2)**, where

$$\omega(t)=s(t)+p(t)+q(t) \quad \textbf{(3)}$$

Taking differentiation of function (3), we get $\frac{d\omega}{dt}=\frac{ds}{dt}+\frac{dp}{dt}+\frac{dq}{dt}$ **(4)**

Replace $\frac{ds}{dt}$, $\frac{dp}{dt}$ and $\frac{dq}{dt}$ mentioned in (2) on **Equation 4**

$$\frac{d\omega}{dt}\leq\left(\frac{-r}{k}s^2+\mu pq+(1+r)s\right)-(s+\delta_1 p+\delta_2 q)\ \leq(1+r)-m\omega \quad \textbf{(5)}$$

From (5) we deduce $\frac{d\omega}{dt}\leq V-m\omega$, Where $m=\min\{1,\delta_1,\delta_2\}$ and $V=1+r$ provided with applying the theory of differential equation

$\omega(t)\leq\frac{1}{m}[V+\omega(t_0)e^{-mt}]$ resulting $\lim_{t\to\infty}\omega(t)\leq\frac{V}{m}$ **(6)**

Hence all the solution of **System 2** on the region $\Omega=\left\{(s,p,q)\in\mathbb{R}^3_+;\ \omega(t)\leq\frac{V}{m}\right\}$ are uniformly bounded.

As a direct computation **System 2** has about five different non-negative equilibrium points, $E_0=(0,0,0)$ (trivial equilibrium point) TEP, $E_1=(1,0,0)=(\bar{p},0,0)$ exist as k=1 (axial

equilibrium point) AEP , the third equilibrium point is $E_2 = (\bar{s}, \bar{p}, 0)$ (top predator free equilibrium point) TPFE exist in the sp −plane if and only if the following conditions are met

$\bar{s} = \frac{\delta_1 a_1}{\gamma_1 - \delta_1}$ and $\bar{p} = \frac{r}{\gamma_1}\left(1 - \frac{\delta_1 a_1}{k(\gamma_1 - \delta_1)}\right)\left(a_1 + \frac{\delta_1 a_1}{\gamma_1 - \delta_1}\right)$ for $\gamma_1 > \delta_1$ **(7)**

The fourth equilibrium point is $E_3 = (\hat{s}, 0, \hat{q})$ (middle predator free equilibrium point) MPFE exists in the sq −plane when the conditions $\hat{s} = k$ and $A = 1 - \frac{a_2 \gamma_2}{e(\gamma_2 - \alpha\delta_2)}$ satisfied.

2.2. Theorem The interior equilibrium point (IEP) exists as an appropriate condition satisfied

Proof. The IEP $E^* = (s^*, p^*, q^*)$ simply can be determined by solving the equations mentioned in (2)

$\varphi_1(s^*, p^*, q^*) = 0, \varphi_2(s^*, p^*, q^*) = 0, \ \varphi_3(s^*, p^*, q^*) = 0$ **(8)**

E^* existed as s^* is a positive solution for the equation

$L_1(s^*)^3 + L_2(s^*)^2 + L_3 s^* + L_4 = 0$ **(9)**

With: $L_1 = \frac{\beta r}{k\gamma_2}(a_2 + \alpha e(1-A) + H_1)$, $L_2 = \frac{\delta_1 \beta r}{k\gamma_2}(a_2 + \alpha e(1-A) + H_1) - \frac{r}{k} - \frac{2\beta r^2}{k\gamma_2}(a_2 + \alpha e(1-A) + H_1)$, $L_3 = r + \beta r^2(a_2 + \alpha e(1-A) + H_1) - \delta_1 \beta r(a_2 + \alpha e(1-A) + H_1) - \frac{a_1 r}{k} - \frac{\delta_1 \beta r}{k\gamma_2}(a_2 + \alpha e(1-A) + H_1)$, $L_4 = a_1 r + \beta r^2 \frac{a_1}{\gamma_2}(a_2 + \alpha e(1-A) + H_1) - \frac{a_1 \beta r^2}{k\gamma_2}(a_2 + \alpha e(1-A) + H_1) - \frac{\delta_1 \beta r}{\gamma_2}(a_2 + \alpha e(1-A) + H_1) - \gamma_1 H_1$, Where

$$\mathrm{H}_1 = \frac{\gamma_2 - \delta_2 - \mu(a_2 + \alpha e(1-A)) + \sqrt{(\gamma_2 - \delta_2 - \mu(a_2 + \alpha e(1-A)))^2 + 4\mu(-\delta_2 a_2 + e(1-A)(\gamma_2 - \delta_2 \alpha))}}{2\mu}$$

The IEP exists only if the below inequality (10) holds.

$(\gamma_2 - \delta_2 - \mu(a_2 + \alpha e(1-A)))^2 > 4\mu(-\delta_2 a_2 + e(1-A)(\gamma_2 - \delta_2 \alpha))$ **(10)**

Equation 8 has three different roots by aid of Descartes' rule of signs [22] when **Condition 11** satisfied: $L_1 > 0, L_2 < 0, L_3 > 0 \ and \ L_4 < 0$ **(11)**

3. Local stability analysis

Now, to set up the local stability, the general Jacobian matrix of **System (2)** denoted by $\mathcal{J}(s,p,q)$ at spq −plane is determined by

$$\mathcal{J}(s,p,q) = \begin{bmatrix} s\frac{\partial \varphi_1}{\partial s} + \varphi_1 & s\frac{\partial \varphi_1}{\partial p} & s\frac{\partial \varphi_1}{\partial q} \\ p\frac{\partial \varphi_2}{\partial s} & p\frac{\partial \varphi_2}{\partial p} + \varphi_2 & p\frac{\partial \varphi_2}{\partial q} \\ q\frac{\partial \varphi_3}{\partial q} & q\frac{\partial \varphi_3}{\partial p} & q\frac{\partial \varphi_3}{\partial q} + \varphi_3 \end{bmatrix} = [\mathcal{X}_{ij}]_{3\times 3} \quad \textbf{(12)}$$

Where $\varphi_1 = \varphi_1(s,p,q)$, $\varphi_2 = \varphi_2(s,p,q)$, $\varphi_3 = \varphi_3(s,p,q)$, defined in **System 2** followed by:

$\mathcal{X}_{11} = \frac{-2rs}{k} + \frac{-\gamma_1 a_1 p}{(a_1+s)^2(1+\beta q)} + r$, $\mathcal{X}_{12} = \frac{-\gamma_1 s}{(a_1+s)(1+\beta q)}$, $\mathcal{X}_{13} = \frac{-\gamma_1 \beta s p}{(a_1+s)(1+\beta q)^2}$

$\mathcal{X}_{21} = \frac{\gamma_1 a_1 p}{(a_1+s)^2(1+\beta q)}$, $\mathcal{X}_{22} = \frac{-\gamma_2 q(a_2+\alpha e(1-A))}{(a_2+\alpha e(1-A)+p)^2} + \frac{\gamma_1 s}{(a_1+s)(1+\beta q)} - \delta_1$, $\mathcal{X}_{23} = \frac{-\gamma_1 \beta s p}{(a_1+s)(1+\beta q)^2} - \frac{\gamma_2 p}{(a_2+\alpha e(1-A)+P)}$, $\mathcal{X}_{31} = 0$, $\mathcal{X}_{32} = \frac{\gamma_2 q(a_2+e(A-1)(\alpha-1))}{(a_2+\alpha e(1-A)+q)^2} - \mu q$, $\mathcal{X}_{33} = \frac{\gamma_2(p+e(1-A))}{(a_2+\alpha e(1-A)+p)} - \mu p - \delta_2$

It is clear that the jacobian matrix of **System 2** at TEP E_0 specified by

$$\mathcal{J}_0 = \mathcal{J}(E_0) = \begin{bmatrix} r & 0 & 0 \\ 0 & -\delta_1 & 0 \\ 0 & 0 & \frac{\gamma_2 e(1-A)}{(a_2+\alpha e(1-A))} - \delta_2 \end{bmatrix} \quad \textbf{(13)}$$

the eigenvalues of $\mathcal{J}_0$ are $\lambda_{0s} = r, \lambda_{0p} = -\delta_1 \ and \ \lambda_{0q} = \frac{\gamma_2 e(1-A)}{(a_2+\alpha e(1-A))} - \delta_2$,clearly E_0 is a saddle point. The jacobian matrix of AEP E_1 is written as

$$\mathcal{J}_1 = \mathcal{J}(E_1) = \begin{bmatrix} -r & \frac{\gamma_1 k}{a_1+k} & 0 \\ 0 & \frac{\gamma_1 k}{a_1+k} - \delta_1 & 0 \\ 0 & 0 & \frac{\gamma_2 e(1-A)}{a_2+\alpha e(1-A)} - \delta_2 \end{bmatrix} \quad \textbf{(14)}$$

evidently the eigenvalue $\lambda_{1s} = -r$ is negative, while $\lambda_{1p} = \frac{\gamma_1 k}{a_1+k} - \delta_1$ and $\lambda_{1q} = \frac{\gamma_2 e(1-A)}{a_2+\alpha e(1-A)} - \delta_2$ are negative as $\delta_1 > \frac{\gamma_1 k}{a_1+k}$ and $\frac{\gamma_2 e(1-A)}{a_2+\alpha e(1-A)} < \delta_2$ (14a)

hence E_1 is locally stable equilibrium point when **Conditions 14a** come true.

The jacobian matrix of TPFE $E_2 = (\bar{s}, \bar{p}, 0)$ is $\mathcal{J}_2 = \mathcal{J}(E_2) = [\sigma_{ij}]_{3\times 3}$ **(15)**

where $\sigma_{11} = \frac{-2r\bar{s}}{k} + \frac{-\gamma_1 a_1 \bar{p}}{(a_1+\bar{s})^2} + r$, $\sigma_{12} = \frac{\gamma_1 \bar{s}}{(a_1+\bar{s})}$, $\sigma_{13} = \frac{\gamma_1 \beta \bar{s}\bar{p}}{(a_1+\bar{s})}$, $\sigma_{21} = \frac{\gamma_1 a_1 \bar{p}}{(a_1+\bar{s})^2}$, $\sigma_{22} = \frac{\gamma_1 \bar{s}}{(a_1+\bar{s})} - \delta_1$, $\sigma_{23} = \frac{-\gamma_1 \beta \bar{s}\bar{p}}{(a_1+\bar{s})} - \frac{\gamma_2 \bar{p}}{(a_2+\alpha e(1-A)+\bar{p})}$, $\sigma_{31} = 0, \sigma_{32} = 0, \sigma_{33} = \frac{\gamma_2(\bar{p}+e(1-A))}{(a_2+\alpha e(1-A)+\bar{p})} - \mu\bar{p} - \delta_2$

The eigenvalues of (16) can be written as the characteristic equation

$$(\sigma_{33} - \xi)[(\sigma_{11} - \xi)(\sigma_{22} - \xi) - \sigma_{21}\sigma_{12}] = 0 \quad \textbf{(16)}$$

From **Equation (16)** we have $\xi_q = \sigma_{33}$, $\xi_{s1} = \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sqrt{\sigma_{22}^2+\sigma_{11}^2-2\sigma_{22}\sigma_{11}+4\sigma_{21}\sigma_{12}}}{2}$, $\xi_{s2} = \frac{\sigma_{11}+\sigma_{22}}{2} - \frac{\sqrt{\sigma_{22}^2+\sigma_{11}^2-2\sigma_{22}\sigma_{11}+4\sigma_{21}\sigma_{12}}}{2}$,

As a result, the equilibrium point E_2 is locally asymptotically stable if $\sigma_{11} < 0, \sigma_{22} < 0$ and $\sigma_{33} < 0$ are hold. **(17)**

Now to show the local stability of E_3, the jacobian matrix of MPFE $E_3 = (\hat{s}, 0, \hat{q})$ is $\mathcal{J}_3 = \mathcal{J}(E_3) = [\mathcal{b}_{ij}]_{3\times 3}$, Where

$$\mathcal{J}_3 = \begin{bmatrix} \frac{-2r\hat{s}}{k} + r & \frac{-\gamma_1 \hat{s}}{(a_1+\hat{s})(1+\beta\hat{q})} & 0 \\ 0 & \frac{-\gamma_1 \hat{s}}{(a_1+\hat{s})(1+\beta\hat{q})} - \frac{\gamma_2 \hat{q}}{(a_2+\alpha e(1-A))^2} - \delta_1 & 0 \\ 0 & \frac{\gamma_2 \hat{q}(a_2+e(1-A)(\alpha-1))}{(a_2+\alpha e(1-A))^2} - \mu\hat{q} & \frac{\gamma_2(e(1-A))}{(a_2+\alpha e(1-A))} - \delta_2 \end{bmatrix} \quad \textbf{(18)}$$

The eigenvalues of (16) can be written as the characteristic equation

$(\mathcal{b}_{11} - \tau)(\mathcal{b}_{22} - \tau)(\mathcal{b}_{33} - \tau) = 0$, hence the MPFE equilibrium point E_3 is locally stable if the condition (19) satisfied

$\mathcal{b}_{11} < 0, \mathcal{b}_{22} < 0$ and $\mathcal{b}_{33} < 0$ **(19)**

3.1. Theorem Suppose that the (interior equilibrium point) IEP $E^*$of **System 2** exists, then it is a locally asymptotically stable when the following sufficient conditions are satisfied

$\frac{2rs^*}{k} + \frac{\gamma_1 a_1 p^*}{(a_1+s^*)^2(1+\beta q^*)} > r$, $\frac{\gamma_2 q^*(a_2+\alpha e(1-A))}{(a_2+\alpha e(1-A)+p^*)^2} + \frac{\gamma_1 s^*}{(a_1+s^*)(1+\beta q^*)} < \delta_1$,

$\frac{\gamma_2(p^*+e(1-A))}{(a_2+\alpha e(1-A)+p^*)} < \mu p^* + \delta_2$, $\frac{\gamma_2 q^*(a_2+e(1-\alpha)(1-A))}{(a_2+\alpha e(1-A)+p^*)^2} > \mu q^*$ **(20)**

Proof. The Jacobian matrix of (IEP) E^* written as follows

$$\mathcal{J}^* = \mathcal{J}(E^*) = [\mathcal{y}_{ij}]_{3\times 3} \quad \textbf{(21)}$$

Where $\mathcal{y}_{11} = \frac{-2rs^*}{k} + \frac{-\gamma_1 a_1 p^*}{(a_1+s^*)^2(1+\beta q^*)} + r$, $\mathcal{y}_{12} = \frac{-\gamma_1 s^*}{(a_1+s^*)(1+\beta q^*)}$, $\mathcal{y}_{13} = \frac{-\gamma_1 \beta s^* p^*}{(a_1+s^*)(1+\beta q^*)^2}$,

$\mathcal{y}_{21} = \frac{\gamma_1 a_1 p^*}{(a_1+s^*)^2(1+\beta q^*)}$, $\mathcal{y}_{22} = \frac{\gamma_2 q^*(a_2+\alpha e(1-A))}{(a_2+\alpha eA+p^*)^2} + \frac{\gamma_1 s^*}{(a_1+s^*)(1+\beta q^*)} - \delta_1$,

$\mathcal{y}_{23} = \frac{-\gamma_1 \beta s^* p^*}{(a_1+s^*)(1+\beta q^*)} - \frac{\gamma_2 p^*}{(a_2+\alpha e(1-A)+p^*)}$, $\mathcal{y}_{31} = 0$, $\mathcal{y}_{32} = \frac{\gamma_2 q^*(a_2+e(1-\alpha)(1-A))}{(a_2+\alpha e(1-A)+p^*)^2} - \mu q^*$,

$\mathcal{y}_{33} = \frac{\gamma_2(p^*+e(1-A))}{(a_2+\alpha e(1-A)+p^*)} - \mu q^* - \delta_2$

The characteristic **Equation** of **21** written as

$$\eta^3 + \Phi_1\eta^2 + \Phi_2\eta + \Phi_3 = 0 \quad \textbf{(22)}$$

Where $\Phi_1 = -(y_{11} + y_{22} + y_{33})$, $\Phi_2 = y_{11}y_{22} + y_{11}y_{33} + y_{22}y_{33} - y_{21}y_{12} - y_{32}y_{23}$, $\Phi_3 = -y_{11}y_{22}y_{33} + y_{11}y_{32}y_{23} + y_{33}y_{12}y_{21}$, and $\Delta = \Phi_1\Phi_2 - \Phi_3$, where $\Delta = -{y^2}_{11}(y_{22} - y_{33}) - {y^2}_{22}(y_{11} + y_{33}) - {y^2}_{33}(y_{22} + y_{11}) + (y_{12}y_{21} + y_{23}y_{32})(y_{11} + y_{22} + y_{33}) - 3y_{11}y_{22}y_{33}$

By using Routh-Hurwitz criterion[23], the roots of **Equation 22** have negative real parts, then (IEP) $E^*$is locally asymptotically stable when Φ_1, Φ_3 and Δ are positive as **Conditions 20** verified. We found that by a computation all constraints are true under.

4. Local Bifurcation Analysis

This section discusses and examines the saddle-node, transcritical, and pitchfork bifurcations around the equilibrium points that **System 2** possessed, employing Sotomayor theorem[24-25] with reason: the presence of non –hyperbolic equilibrium points is necessary but not adequate condition for bifurcation. A parameter that causes the Jacobian matrix to have a zero real part eigenvalue will be used as a nominee bifurcation parameter, as proven in the theorems below. **System (2)** must be rewritten as $\frac{d\mathcal{S}}{dt} = F(\mathcal{S})$, where $\mathcal{S} = (s,p,q)^T$and $F(\mathcal{S}) = (s\varphi_1, p\varphi_2, q\varphi_3)^T = (f_1, f_2, f_3)^T$. The second directional derivative of the Jacobian matrix $\mathcal{J}(s,p,q) = DF$ can thus be represented as follows for the general vector $V = (v_1, v_2, v_3)^T$: $D^2F(V,V) = [\mathcal{U}_{i1}]_{3\times1}$ (23)

Where: $\mathcal{U}_{11} = \left(\frac{-2r}{k} + \frac{2a_1\gamma_1 p(a_1+s)}{(a_1+s)^4(1+\beta q)}\right)v_1^2 - \left(\frac{2a_1\gamma_1}{(a_1+s)^2(1+\beta q)}\right)v_1v_2 + \left(\frac{2a_1\gamma_1\beta p}{(a_1+s)^2(1+\beta q)^2}\right)v_1v_3 + \left(\frac{2\beta\gamma_1 s}{(a_1+s)(1+\beta q)^2}\right)v_2v_3 - \left(\frac{2\beta^2\gamma_1 sp(1+\beta q)}{(a_1+s)(1+\beta q)^4}\right)v_3^2$, $\mathcal{U}_{21} = \frac{-2\gamma_1 a_1 p(a_1+s)}{(a_1+s)^4(1+\beta q)}v_1^2 + \left(\frac{2\gamma_1 a_1}{(a_1+s)^2(1+\beta q)}\right)v_1v_2 - \left(\frac{2\gamma_1 a_1\beta p}{(a_1+s)^2(1+\beta q)^2}\right)v_1v_3 + \left[\frac{-\gamma_2 q(a_2+\alpha e(1-A)+p)^2 - 2\gamma_2 qp(a_2+\alpha e(1-A)+p)}{(a_2+\alpha e(1-A)+p)^4} + \frac{\gamma_2 q}{(a_2+\alpha e(1-A)+p)^2}\right]v_2^2 - \left[\frac{\gamma_1\beta s}{(a_1+s)(1+\beta q)} + \frac{\gamma_1\beta s}{(a_1+s)(1+\beta q)^2} + \frac{\gamma_2(a_2+\alpha e(1-A)+2p)}{(a_2+\alpha e(1-A)+p)^2} + \frac{\gamma_2(2p+\alpha e(1-A))}{(a_2+\alpha e(1-A)+p)^2}\right]v_2v_3 + \left(\frac{\gamma_1\beta sp}{(a_1+s)(1+\beta q)^2}\right)v_3^2$

$\mathcal{U}_{31} = \frac{-2\gamma_2 q(a_2+\alpha e(1-A)+p)}{(a_2+\alpha e(1-A)+p)^4}v_2^2 + \left(\frac{2(\gamma_2 a_2 + \alpha e\gamma_2(1-A) - \gamma_2 e(1-A))}{(a_2+\alpha e(1-A)+p)^2} - 2\mu\right)v_3v_2$

4.1. Theorem If **Condition 14a** come true then the (axial equilibrium point) AEP E_1 has no saddle-node nor transcritical bifurcation happened either no pitchfork bifurcation within the parameter $\delta_1 = \delta_1^* = \frac{k\gamma_1}{a_1+k}$.

Proof. The jacobian matrix $\mathcal{J}_1$ in **Equation 14** can be written as

$$D(E_1,\delta_1) = \begin{bmatrix} -r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\gamma_2 e(1-A)}{a_2+\alpha e(1-A)} - \delta_2 \end{bmatrix} \quad \textbf{(24)}$$

$\mathcal{J}_1$has two negative real parts under condition (13a), $\lambda_s = -r$ and $\lambda_q = \frac{\gamma_2 e(1-A)}{a_2+\alpha e(1-A)} - \delta_2$.

Let $V = (v_1, v_2, v_3)^T$ be the eigenvector of $D(E_1,\delta_1)$ corresponding to the eigenvalue $\lambda_p = 0$, such that $D(E_1,\delta_1)\,V = \lambda_p V$,thus we obtain that $V = (0, v_2, 0)^T$, where $v_2 \neq 0$, $v_2 \in \mathbb{R}$. Let $\widehat{W} = (\widehat{w}_1, \widehat{w}_2, \widehat{w}_3)^T$be the eigenvector of $D(E_1,\delta_1)^T$ associated with eigenvalue $\lambda_p = 0$ to acquire $\widehat{W} = (0, \widehat{w}_2, 0)^T$, where $\widehat{w}_2 \neq 0$ and $\widehat{w}_2 \in \mathbb{R}$. Since $\frac{\partial}{\partial\delta_1}F(E_1) = (0, -p, 0)^T$ where $F = (f_1, f_2, f_3)^T$ as a result $F_{\delta_1}(E_1,\delta_1^*) = (0,0,0)^T$

and for this $\widehat{W}^T F_{\delta_1}(E_1,\delta_1^*) = 0$, an immediate counting shows that $\frac{\partial}{\partial\delta_1}D(E_1,\delta_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = DF(E_1,\delta_1^*)$, then $\widehat{W}^T DF(E_1,\delta_1^*)V = -v_2\widehat{w}_2 \neq 0$ **(25)**

we observe from **Equation 22** that $D^2F(E_1,\delta_1)(V,V) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ **(26)**

Hence $\widehat{W}^T D^2F(E_1,\delta_1)\,(V,V) = 0$ **(27)**

Based on the conditions of Sotomayor's theorem, we conclude that from **Equations 25** and **27** the saddle-node bifurcation is not achieved and the conditions for the transcritical bifurcation are not met. Finally, as a result, from **Equation 26** the pitchfork bifurcation does not exist.

4.2.Theorem If **Condition 17** hold, then the (top predator free equilibrium point) TPFE E_2 has a saddle-node bifurcation but neither transcritical bifurcation nor pitchfork bifurcation, around the parameter $\delta_2 = \delta_2^* = \frac{\gamma_2(\bar{p}+e(1-A))}{(a_2+\alpha e(1-A)+\bar{p})} - \mu\bar{p}$.

Proof. We can rewrite the jacobian matrix $\mathcal{J}_2$ in **Equation 15** as

$$D(E_2,\delta_2^*) = \begin{bmatrix} \frac{-2r\bar{s}}{k} + \frac{-\gamma_1 a_1\bar{p}}{(a_1+\bar{s})^2} + r & \frac{-\gamma_1\bar{s}}{(a_1+\bar{s})} & \frac{\gamma_1\beta\bar{s}\bar{p}}{(a_1+\bar{s})} \\ \frac{\gamma_1 a_1\bar{p}}{(a_1+\bar{s})^2} & \frac{\gamma_1\bar{s}}{(a_1+\bar{s})} - \delta_1 & \frac{-\gamma_1\beta\bar{s}\bar{p}}{(a_1+\bar{s})} - \frac{\gamma_2\bar{p}}{(a_2+\alpha e(1-A)+\bar{p})} \\ 0 & 0 & 0 \end{bmatrix} \quad \textbf{(28)}$$

under **Condition 18,** $D(E_2,\delta_2^*)$ has two real negative eigenvalues $\xi_s = \frac{-2r\bar{s}}{k} + \frac{-\gamma_1 a_1\bar{p}}{(a_1+\bar{s})^2} + r$ and $\xi_p = \frac{\gamma_1\bar{s}}{(a_1+\bar{s})} - \delta_1$. Let $\bar{V} = (\bar{v}_1,\bar{v}_2,\bar{v}_3)^T$ be the eigenvector of $D(E_2,\delta_2^*)$ conformable to the eigenvalue $\xi_q = 0$, consequently $\bar{V} = (K_1\bar{v}_2,\bar{v}_2,K_2\bar{v}_2)^T$, where

$K_1 = \frac{\left(\frac{-\gamma_1\bar{s}}{(a_1+\bar{s})}\right)\left(\frac{-\gamma_1\beta\bar{s}\bar{p}}{(a_1+\bar{s})} - \frac{\gamma_2\bar{p}}{(a_2+\alpha e(1-A)+\bar{p})}\right) - \left(\frac{\gamma_1\bar{s}}{(a_1+\bar{s})} - \delta_1\right)\left(\frac{\gamma_1\beta\bar{s}\bar{p}}{(a_1+\bar{s})}\right)}{\left(\frac{-2r\bar{s}}{k} + \frac{-\gamma_1 a_1\bar{p}}{(a_1+\bar{s})^2} + r\right)\left(\frac{-\gamma_1\beta\bar{s}\bar{p}}{(a_1+\bar{s})} - \frac{\gamma_2\bar{p}}{(a_2+\alpha e(1-A)+\bar{p})}\right) + \left(\frac{\gamma_1 a_1\bar{p}}{(a_1+\bar{s})^2}\right)\left(\frac{\gamma_1\beta\bar{s}\bar{p}}{(a_1+\bar{s})}\right)}$, $K_2 = \frac{K_1\left(\frac{2r\bar{s}}{k} + \frac{\gamma_1 a_1\bar{p}}{(a_1+\bar{s})^2} - r\right) + \frac{-\gamma_1\bar{s}}{(a_1+\bar{s})}}{\frac{\gamma_1\beta\bar{s}\bar{p}}{(a_1+\bar{s})}}$

And $\bar{v}_2 \neq 0$. Let $\mathfrak{U} = (u_1,u_2,u_3)^T$ represent the eigenvector of $D(E_2,\delta_2^*)^T$ corresponding to the eigenvalue $\xi_q = 0$, then we conclude $\mathfrak{U} = (0,0,u_3)^T$, where $u_3 \neq 0$ and $u_3 \in \mathbb{R}$.

Since $\frac{\partial}{\partial\delta_2}F(E_2,\delta_2) = (0,0,-q)^T$as a result $F_{\delta_2}(E_2,\delta_2^*) = (0,0,-1)^T$ this takes us to obtain that $\mathfrak{U}^T F_{\delta_2}(E_2,\delta_2^*) = -u_3 \neq 0$, Also, direct computation shows that:

$\frac{\partial}{\partial\delta_2}D(E_2,\delta_2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, thus **Equation 2** turns that $\mathfrak{U}^T D^2F(E_2,\delta_2)\,(\bar{V},\bar{V}) = u_3K_2\bar{v}_2^2\left(\frac{2(\gamma_2 a_2+\alpha e\gamma_2(1-A))}{(a_2+\alpha e(1-A)+p)^2} - \mu\right) \neq 0$, depending on Sotomayor theorem deduced that E_2 has a Saddle-node bifurcation, but no transcritical bifurcation, we can easily prove that $\mathfrak{U}^T D^3F(E_2,\delta_2)\,(\bar{V},\bar{V},\bar{V}) \neq 0$ to confirm E_2has no pitchfork bifurcation.

4.3.Theorem The (interior equilibrium point) IEP satisfied a saddle –node bifurcation through the parameter $r = r^*$, assuming that **Condition 20** undergoes around E^*, provided the following conditions satisfied

$$\frac{\gamma_2(p+e(1-A))}{(a_2+\alpha e(1-A)+p)} > \mu p + \delta_2,\ (\mathcal{Z}_3\mathcal{U}_{11}^* + \mathcal{Z}_4\mathcal{U}_{21}^* + \mathcal{U}_{31}^*) \neq 0 \quad \textbf{(29)}$$

Where $r = r^* = \frac{\gamma_1 a_1 p}{(a_1+s)^2(1+\beta q)(1-\frac{2s}{k})} + \frac{\mathcal{Y}_{21}\mathcal{Y}_{12}\mathcal{Y}_{33} - \mathcal{Y}_{21}\mathcal{Y}_{13}\mathcal{Y}_{23}}{(1-\frac{2s}{k})(\mathcal{Y}_{22}\mathcal{Y}_{33} - \mathcal{Y}_{23}\mathcal{Y}_{32})}$

Proof. We define the jacobian $\mathcal{J}^* = D(E^*,r^*) = \mathcal{J}(E^*) = \left[\mathcal{Y}_{ij}\right]_{3\times3}$ which given in **Equation 21**, by using **Condition 29** we compute that $|D(E^*,r^*)| = 0$ for this the jacobian matrix has a zero eigenvalue λ^* to make the equilibrium point E^* a non-hyperbolic point. Let $\mathcal{v}^* = (\mathcal{v}_1^*,\mathcal{v}_2^*,\mathcal{v}_3^*)^T$ be the eigenvector of $D(E^*,r^*)$ corresponding to $\lambda^* = 0$, where $(E^*,r^*)\mathcal{v}^* = \lambda^*\mathcal{v}^*$, to obtain $\mathcal{v}^* = (\mathcal{Z}_1\mathcal{v}_3^*,\mathcal{Z}_2\mathcal{v}_3^*,\mathcal{v}_3^*)^T$, such that $\mathcal{v}_3^* \neq 0$, $\mathcal{v}_3^* \in \mathbb{R}$ and $\mathcal{Z}_1 = \frac{\mathcal{Y}_{22}\mathcal{Y}_{33}}{\mathcal{Y}_{23}\mathcal{Y}_{32}} - \frac{\mathcal{Y}_{23}}{\mathcal{Y}_{21}}$, $\mathcal{Z}_2 = \frac{-\mathcal{Y}_{33}}{\mathcal{Y}_{32}}$, let $\mathfrak{Z} = (\mathfrak{Z}_1^*,\mathfrak{Z}_2^*,\mathfrak{Z}_3^*)^T$ be the eigenvector associated to $D(E^*,r^*)^T$ associated with the eigenvalue $\lambda^* = 0$, which gives $\mathfrak{Z} = (\mathcal{Z}_3\mathfrak{Z}_3^*,\mathcal{Z}_4\mathfrak{Z}_3^*,\mathfrak{Z}_3^*)^T$, for $\mathfrak{Z}_3^* \neq 0$, $\mathfrak{Z}_3^* \in \mathbb{R}$ and $\mathcal{Z}_3 = \frac{-\mathcal{Y}_{21}}{\mathcal{Y}_{11}}$, $\mathcal{Z}_4 = \frac{(\mathcal{Y}_{32}+\mathcal{Y}_{33})\mathcal{Y}_{11}}{-(\mathcal{Y}_{22}+\mathcal{Y}_{23})\mathcal{Y}_{11}+(\mathcal{Y}_{12}+\mathcal{Y}_{13})\mathcal{Y}_{21}}$, next we get $\mathfrak{Z}^T\frac{\partial F}{\partial r} = \mathcal{Z}_3(s-\frac{s}{k})\mathfrak{Z}_3^* \neq 0$, where $\frac{\partial F}{\partial r} = (s-\frac{s}{k},0,0)$ for $F = (f_1,f_2,f_3)$ corresponding to **System 2**

Hence, the first condition in the concept of Saddle- node bifurcation is convinced. Now, according to **Equation 21**, it is gained that $D^2F(E^*,r^*)(V,V) = [\mathcal{U}^*{}_{i1}]_{3\times1}$

Where:

$$\mathcal{U}^*{}_{11} = \left(\frac{-2r}{k} + \frac{2a_1\gamma_1 p^*(a_1+s^*)}{(a_1+s^*)^4(1+\beta q^*)}\right)\mathcal{Z}_1^2 v_3^{*2} - \left(\frac{2a_1\gamma_1}{(a_1+s^*)^2(1+\beta q^*)}\right)\mathcal{Z}_1\mathcal{Z}_2 v_3^{*2} + \left(\frac{2a_1\gamma_1\beta p^*}{(a_1+s^*)^2(1+\beta q^*)^2}\right)\mathcal{Z}_1 v_3^{*2} + \left(\frac{2\beta\gamma_1 s^*}{(a_1+s^*)(1+\beta q^*)^2}\right)\mathcal{Z}_2 v_3^{*2} - \left(\frac{2\beta^2\gamma_1 s^* p^*(1+\beta q^*)}{(a_1+s^*)(1+\beta q^*)^4}\right)v_3^{*2}$$

$$\mathcal{U}^*{}_{21} = \frac{-2\gamma_1 a_1 p^*(a_1+s^*)}{(a_1+s^*)^4(1+\beta q^*)}\mathcal{Z}_1^2 v_3^{*2} + \left(\frac{2\gamma_1 a_1}{(a_1+s^*)^2(1+\beta q^*)}\right)\mathcal{Z}_1 v_3^{*2} - \left(\frac{2\gamma_1 a_1\beta p^*}{(a_1+s^*)^2(1+\beta q^*)^2}\right)\mathcal{Z}_1 v_3^{*2} + \left[\frac{-\gamma_2 q^*(a_2+\alpha e(1-A)+p^*)^2+2\gamma_2 q^* p^*(a_2+\alpha e(1-A)+p^*)}{(a_2+\alpha e(1-A)+p^*)^4} + \frac{\gamma_2 q^*}{(a_2+\alpha e(1-A)+p^*)^2}\right]\mathcal{Z}_2 v_3^{*2} - \left[\frac{2\gamma_1\beta s^*}{(a_1+s^*)(1+\beta q^*)} + \frac{2\gamma_2(a_2+\alpha e(1-A)+p^*)}{(a_2+\alpha e(1-A)+p^*)^2}\right]\mathcal{Z}_2 v^{*2}{}_3 + \left(\frac{\gamma_1\beta s^* p^*}{(a_1+s^*)(1+\beta q^*)^2}\right)v_3^{*2}$$

$$\mathcal{U}^*{}_{31} = \frac{-2\gamma_2 q^*(a_2+\alpha e(1-A)+p^*)}{(a_2+\alpha e(1-A)+p^*)^4}\mathcal{Z}_2^2 v_3^{*2} + \left(\frac{2(\gamma_2 a_2+\alpha e\gamma_2(1-A))}{(a_2+\alpha e(1-A)+p^*)^2} - \mu\right)\mathcal{Z}_2 v_3^{*2}$$

Therefore, using **Condition 29**; it follows that $\mathfrak{Z}^T D^2F(E^*, r^*)(V,V) = (\mathcal{Z}_3\mathcal{U}^*_{11} + \mathcal{Z}_4\mathcal{U}^*_{21} + \mathcal{U}^*_{31})\mathfrak{Z}^*_3 \neq 0$. Hence, **System 2** acquires a Saddle –node bifurcation close to E^* when $r = r^*$.

4.4. Hopf – Bifurcation

In this section the prospect of Hopf-Bifurcation is discussed near the (interior equilibrium point) IEP $E^*$of **System 2**.

4.4.1Theorem The interior equilibrium point IEP E^* of **System 2** has Hopf-Bifurcation on the parameter $\gamma_2 = \gamma_2^*$, when the following conditions come true

$\Phi_1\Phi_2 < \Phi_3$, $\gamma_2^* = \delta_2 + \mu p^* - \mathbb{Q}$

Where

$$\mathbb{Q} = \frac{\left(a_{11}^2+a_{22}^2-(a_{23}+a_{32})+\sqrt{\left(a_{11}^2+a_{22}^2-(a_{23}+a_{32})\right)^2-4(a_{11}+a_{22})(a_2+\alpha e(1-A)+p)\left((a_{21}a_{12}(a_{11}+a_{22})-a_{11}^2a_{22}-a_{22}^2a_{11}+a_{32}a_{23}a_{22}\right)}\right)}{2(p+e(1-A))(a_{11}+a_{22})}$$

Proof. Let us call back the jacobian matrix j^* and its characteristic **Equation 22**, regarding that $\Phi_1\Phi_2 - \Phi_3 = \Delta$ if $\Delta = 0 \implies \Phi_1\Phi_2 = \Phi_3$ and when $\gamma_2 = \gamma_2^*$ hence the characteristic equation would be

$$(t^2 + \Phi_2)(t + \Phi_1) = 0 \qquad \textbf{(30)}$$

Consequently, we obtain that $t = \Phi_1$ and $t_{2,3} = \mp i\sqrt{\Phi_2} = \pm i\rho_2(\gamma_2)$ evidently $\Phi_1 > 0$ and $\Phi_2 > 0$ due to **Condition 22**, for this the jacobian matrix have three eigenvalues the first one is negative real eigenvalue and two pure imaginary complex. Therefore ,the first standars for having Hopf-bifurcation satisfied over and above, where γ_2 belongs to neighborhood of γ_2^* then the complex conjugate eigenvalues become $t_{2,3} = \rho_1(\gamma_2) \pm i\rho_2(\gamma_2)$, next substituting $t_{2,3}$ in **(20)** and take derivative of the resulting equation with respect to γ_2, then a comparison between both of the real and imaginary parts for the resulting two sides to acquire

$$\mathfrak{y}\rho_1' - \Theta\rho_2' = -\Upsilon\,,\ \Theta\rho_1' - \mathfrak{y}\rho_2' = -\Gamma \qquad \textbf{(31)}$$

Where $\mathfrak{y} = 3\rho_1^2 + 2\Phi_1\rho_1 - 3\rho_2^2 + \Phi_2$, $\Theta = 6\rho_1\rho_2 + 2\Phi_1\rho_2$, $\Upsilon = \Phi_1'\rho_1^2 - \Phi_1'\rho_2^2 + \Phi_2'\rho_1 + \Phi_3'$, $\Gamma = 2\Phi_1'\rho_1\rho_2 + \Phi_2'\rho_2$. Solving the linear system **(31)** for ρ_1, ρ_2 gives`

$\rho_1^2 = \frac{\mathfrak{y}\Upsilon+\Gamma\Theta}{\mathfrak{y}^2+\Theta^2}$, $\rho_2^2 = \frac{-\Gamma\mathfrak{y}+\Upsilon\Theta}{\mathfrak{y}^2+\Theta^2}$. Wherefore the proof is done if and only if $\rho_1 \neq 0$ and $\mathfrak{y}\Upsilon + \Gamma\Theta \neq 0$, however $\rho_1 = 0$ and $\rho_2 = \sqrt{\Phi_2}$ produce that $\mathfrak{y} = -2\Phi_2$, $\Theta = -2\Phi_1\sqrt{\Phi_2}$, $\Upsilon = -\Phi_1'\Phi_2 + \Phi_3'$ and $\Gamma = \Phi_2'\sqrt{\Phi_2}$, it is augured that $\mathfrak{y}\Upsilon + \Gamma\Theta = -2\,\Phi_2\Phi_3 \neq 0$, thus, **System 2** has Hopf-bifurcation around q under condition (30) on the parameter $\gamma_2 = \gamma_2^*$.

4.5 Numerical Analysis

In this section, we manner Runge-Kutta numerical simulation[25] using MATLAB software to state the analysis feedback disputed later. An output **System 2** can be solved for different sets of parameters and initial conditions, which adopt different biological possibilities. The analysis and numerical representation are divided into two groups of images based on a set of data of parameters. The first is concerned with the role of changing the system's parameter on its

asymptotic performance found in Figures 1-3, while the second group presents bifurcation behaviour around the interior equilibrium point E* with respect to prey and top predator directions as shown in Figures 4 and 5, respectively. First of all, Figure 1 ensures theoretical findings related to the presence of local stability cohabitation to the equilibrium point E* under proven conditions for a set of data **Parameters 31**:

$$r = 0.6, a_1 = 0.6, a_2 = 0.6, k = 0.5, A = 0.1, \gamma_1 = 0.8, \gamma_2 = 0.9, \delta_1 = 0.15, \delta_2 = 0.11,$$
$$\mu = 0.5, e = 0.01, \beta = 0.3, \alpha = 0.5 \quad \textbf{(31)}$$

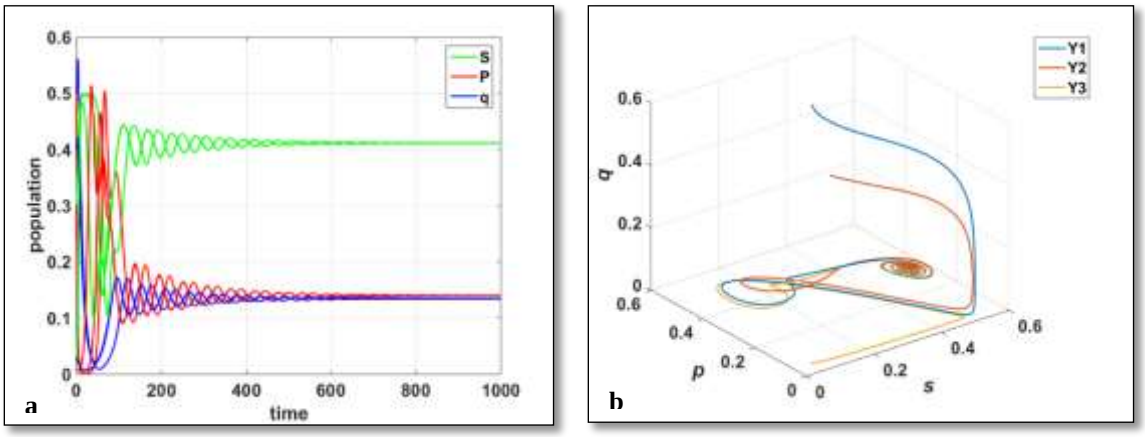


Figure 1. Trajectory of system (2) for data in (31) starting from different initial points Y1= (0.4,0.5,0.5), Y2= (0.3, 0.3, 0.4), Y3 = (0.02, 0.01, 0.03), a. time series for data (31), b. 3D portrait of stable attractor for data (31).

It is clear **Figure 1a** illustrates time series trajectories of **System 2**, while **Figure 1b** shows local stability near the interior equilibrium point E* (0.4112, 0.1397, 0.1384) in the real plane with three different initial points Y1 = (0.4, 0.5, 0.5), Y2 = (0.3, 0.3, 0.4), and Y3 = (0.02, 0.01, 0.03). The first case is to state the stability of **System 2** is through modifying the additional food parameters and fear parameter in (31), keeping the rest of the data fixed. We noticed there is a marked reduction of the top predator to 0.1233, returning to the effect of increasing A = 1, coinciding with the fear parameter of the middle predator = 0 in **Figure 2b**; the top predator increases and arrives at 0.1439.

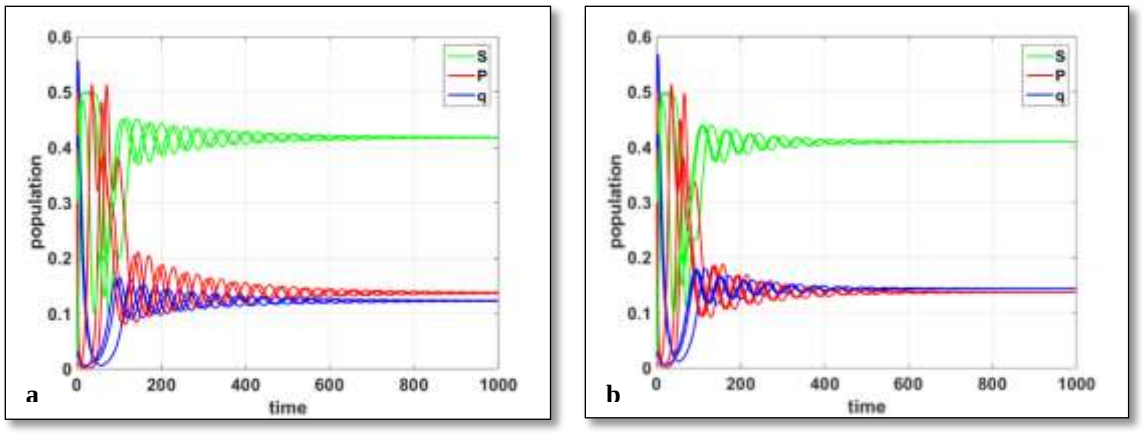


Figure 2. Time series trajectory of system (2) for data in (31) starting from different initial points Y1,Y2 and Y3 varying A= 0 to A= 1 and $\beta = 0.8$ to β =0.01 in a. and b. respectively.

As per the representative presented by **Figure 3a** the trajectories of **System 2** move asymptotically to the top predator free equilibrium point E3 as $\gamma_2 \leq 0.75$ for the set (31), in Figure. 3b it is noticed that the paths of system (2) return to middle predator free equilibrium point E2 with $e \leq 0.07$ for the data (31).

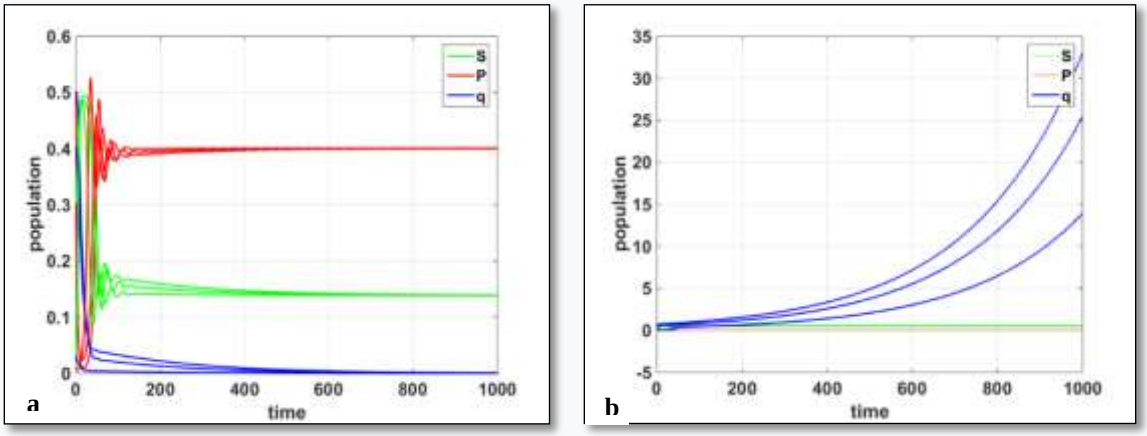


Figure 3. The trajectories of system (2) with set of data (31) (a) As $\gamma_2 = 0.75$, (b)With $e = 0.07$.

The second case shown in **Figure 4a** and **b** ensures the saddle-node bifurcation behavior of the interior equilibrium point E^* established in **Theorem 4.3.** to clarify an occurrence of prey population with respect to the parameter $r \in (0,1)$. Evidence in **(20)** and **(29)** at **Theorem 3.1** and **Theorem 4.3** shows these bifurcations presented for the initial points Z = (0.4,0.5,0.3), with the data set (32): $a_1 = 0.2, a_2 = 0.1,\ k = 0.7, A = 0.7, \gamma_1 = 0.5, \gamma_2 = 0.2, \delta_1 = 0.2, \delta_2 = 0.5,$ $\mu = 0.6, e = 0.3,\ \beta = 0.7, \alpha = 0.3$ **(32)**

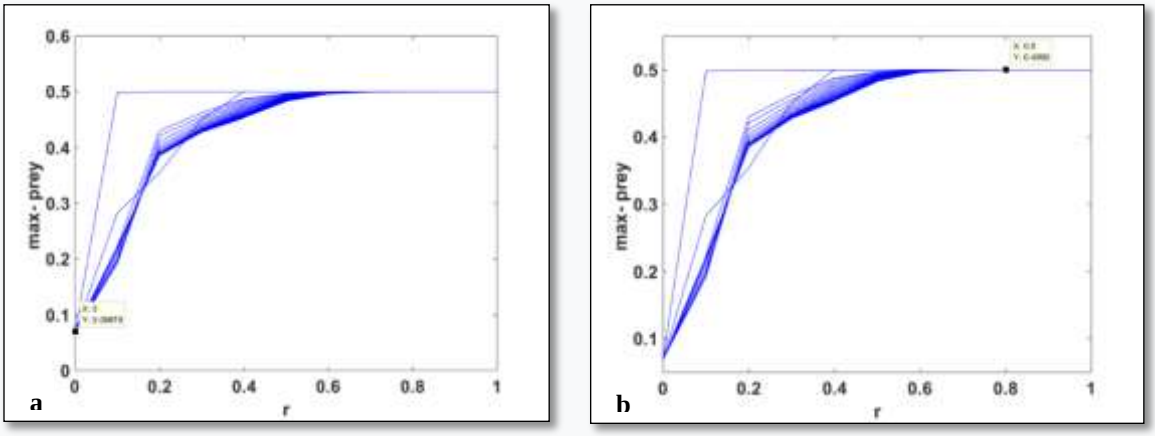


Figure 4. a. and b. validated the saddle- node bifurcation around the interior equilibrium point E^* according to data in (32) with respect to the parameter r coupled with initial point = (0.4,0.5,0.3).

Obviously, the prey density inverted from an unstable state at 0.0697, generating diverging paths to a stable state at point 0.8, forming a saddle- node bifurcation. The next representation (**Figure 5**) is about Hopf bifurcation of the maximum top predator population with respect to the parameter γ_2 according to data set **33**

$\delta_1 = 0.2, \delta_2 = 0.5,\ a_1 = 0.4, a_2 = 0.2, k = 0.6, A = 0.7\ , \gamma_1 = 0.1, r = 0.3, \beta = 0.7, \alpha =$ $0.3, e = 0.3, \mu = 0.6$ **(33)**

It is seen that when the aggression rate $\gamma_2 \in (0\ , 0.4)$ then **System 2** shows a stable steady state for the top predator population; the intense oscillation of the top predator population occurs as $\gamma_2 \in (0.45,0.5)$, which indicates a Hopf- bifurcation happening, as $\gamma_2 > 0.5$ the oscillation of the prey population becomes clearer and more complicated, which explain the dynamical behavior of **System 1**.

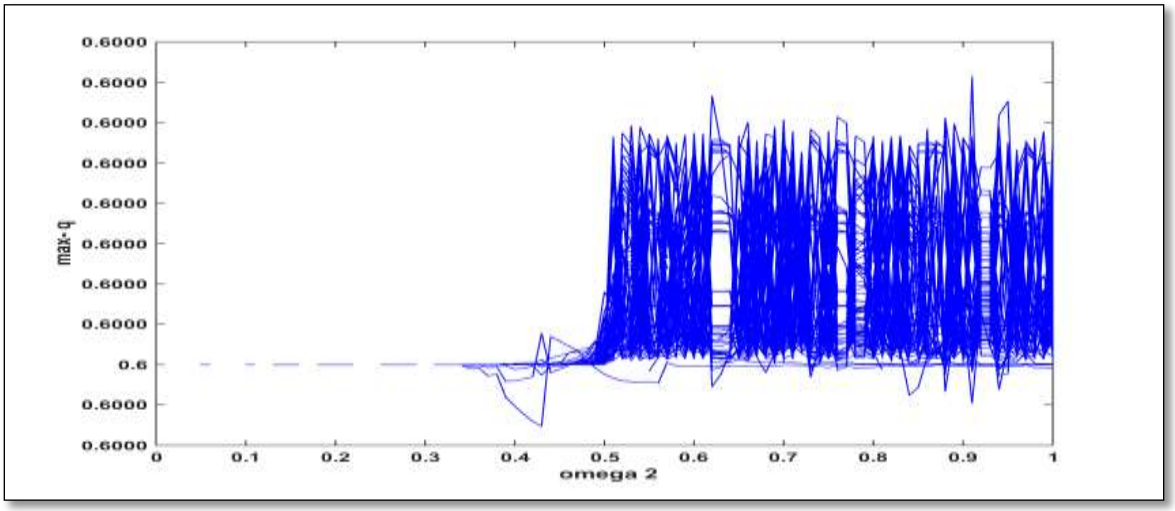


Figure 5. Hopf-Bifurcation of E^* for prey population with respect to the parameter γ_2 with initial point *(s , p , q)* = (0.4,0.5,0.3) and data set (33).

From the above, we arrive that any change in the parameters of additional food, fear presented in **System 2** causes modulation in stability levels of competing communities within the food chain with respect to different initial points. In the other hand the bifurcations behavior of **System 2** are affected by the parameters r and γ_2.

5. Conclusion

An analysis of bifurcation for three species food systems (prey-middle predator-top predator) is mathematically formulated and studied, considering the middle predator fearing the top predator. The effect is clearly visible on studying system stability, which concurs that the top predator has an additional food and responds with anti-predatory behaviour. The solutions of existence and local stability properties are studied. There are only five non-negative equilibrium points in the system; four of them are locally stable. A numerical simulation was used to examine local dynamics and determine the impact of changing parameters, in particular fear and additional food, on the population size of the top predator community shown in Figure 1. and Figure 2, successively. While Figure 3 demonstrates the impact of decreasing the parameter (top predator aggression rate) along with increasing the conversion rate. Ultimately the saddle node and additionally the Hopf bifurcation are thoroughly explored around the interior equilibrium point according to the prey (r (growth rate)) and the top predator ((top predator aggression rate)), figured in Figure 4. and Figure 5, respectively. As a recommendation, there is a possibility to generalise the study for the system under investigation to different selected functional responses in the presence of other influences, such as harvesting or refuge.

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Conflict of Interest

The author declares no conflict of interest.

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