Ibn Al-Haitham Journal for Pure and Applied Sciences
Journal homepage: jih.uobaghdad.edu.iq
PISSN: 1609-4042, EISSN: 2521-3407
IHJPAS. 2026; 39(3)

Qualitative Properties to Type of Second Order Emden-Fowler Dynamic Equation

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Received: 25/January/2026
Accepted: 23/April/2026
Published: 20/July/2026
doi.org/10.30526/39.3.4436

Abstract
This study aimed to secure oscillation properties with asymptotic natures to solutions of Emden-Fowler dynamic equations on a time scale. The primary objectives were to establish new sufficient criteria that guarantee either the oscillatory of solutions or properties of non-oscillatory solutions in mean of asymptotic behavior, such as convergence to zero or to infinity. The research was conducted within the unified framework of difference equations. The classification principle for non-oscillatory solutions, along with the use of generalized and fundamental lemmas, was employed to derive the desired properties, including oscillation criteria and the analysis of asymptotic behavior. The analysis yielded new oscillation criteria with sufficient conditions. For the non-oscillatory case, the study provided explicit conditions under which every solution must eventually satisfy a specific asymptotic property, an illustrative example for application of new results with an effective condition.
It was concluded that the newly established criteria are both effective and widely applicable, as they extend and generalize many known results for differential and difference equations. This work provides a comprehensive theoretical framework for understanding the qualitative phenomena of this significant class of dynamic equations.
Keywords: Emden-Fowler dynamic equation, Neutral type Second Order, Asymptotic Behavior, Oscillation Property, Difference Equation.

1. Introduction

The theory of delay equations has diverse areas of study and scientific research, such as consideration of the solutions[1–4] and consideration of qualitative properties such as stability[5,6] and oscillatory and asymptotic properties[7–9]. In particular, the theory of dynamic equations on time scales, pioneered in 1988[10], provides a powerful unified framework for studying differential equations (continuous) and difference equations (discrete) simultaneously[11–13]. This calculus seamlessly bridges the split between discrete and continuous analysis, allowing for the exploration of systems that evolve in a hybrid or non-uniform time domain. Within this expansive theory, the Emden-Fowler equation stands as a cornerstone mode[14–16]. Originally formulated to describe the gravitational potential of a gaseous sphere in astrophysics, this nonlinear equation has since found profound relevance in fluid mechanics, nuclear physics, and chemical reactions[17,18].

Understanding the oscillatory properties (whether solutions oscillate around an equilibrium) and asymptotic behavior (the long-term trend of solutions as time approaches infinity) of such equations is not merely a mathematical exercise but a necessity for predicting system stability[6].

The specific purpose of this research is to conduct a qualitative analysis of the second-order dynamic Emden-Fowler equation[19]. We aim to establish sufficient conditions that completely determine the nature of its solutions. This involves creating a definitive prediction of whether all solutions will oscillate or if non-oscillatory solutions exist and, if so, what their ultimate fate will

be. The methodology is based on several steps to consider qualitative properties of the Emden–Fowler dynamic equation on time scales as follows:
Step1: Problem formulation.
Step2: Preliminaries and definitions.
Step3: Define suitable auxiliary functions and coefficients.
Step4: Establish oscillation conditions as in the first theorem.
Step5: Establish asymptotic behavior conditions as in the second theorem.
Step6: Illustrative Examples to demonstrate oscillation criteria and asymptotic behavior.
Step7: Conclusions.

2. Materials and Methods

In the present work new sufficient conditions are present for second-order neutral difference equations:

$$\Delta^2\left(Y_n + p_n Y_{n-\mu}\right) + q_n Y^{\zeta}_{n-\kappa} = 0, \qquad \textbf{(1)}$$

Where ζ is positive integer number, p_n, q_n are two sequences of real numbers.
Y_n represents the solution of the **Equation (1)**
And Δ is the forward difference operator defined as:

$$\Delta Y_n = Y_{n+1} - Y_n \text{ and } \Delta^2 Y_n = \Delta(\Delta Y_n) \qquad \textbf{(2)}$$

2.1. Definition

A regular solution Y_n of **Equation (1)** is called to have oscillatory property if the terms of a sequence Y_n are neither eventually negative nor eventually positive, otherwise, it's a nonoscillatory[20].

2.2. Lemma

Suppose that having a real sequence Y_n $n \in N$. Then the subsequent statements hold:
i-Let $\lim_{n\to\infty} \Delta^{\mu} Y_n > 0$ then $\Delta^{i-1} Y_n \to \infty$. With $i = \mu, \mu-1, \ldots, 1$
ii-Let $\lim_{n\to\infty} \Delta^{\mu} Y_n < 0$ then $\Delta^{i-1} Y_n \to \infty$.with $i = \mu, \mu-1, \ldots, 1$[21].

2.3. Lemma

Let $\{F_n\}, \{Y_n\}, \{p_n\}$ be a sequence of real numbers defined for $n \geq n_0 > 0$, $\ni F_n = Y_n + p_n Y_{n-\mu}$, $n \geq n_1$. Where $\mu \geq 0, \mu \in \mathbb{Z}$. Let $p_1, p_2, p_3, p_4 \in \mathbb{R}$,
With following possibilities for p_n:

i. $-1 < -p_n \leq p_n \leq 0$
ii. $0 \leq p_n \leq p_2$
iii. $-p_3 \leq p_n \leq -p_4 < 1$
iv. Let $Y_n > 0, n \geq n_0$, $\lim_{n\to\infty} \inf Y_n = 0$, $\lim_{n\to\infty} F_n = \gamma$, than $\gamma = 0$[22].

3. Results

This section includes two results for obtaining oscillation and asymptotic properties for **Equation (1).**

3.1.Theorem: Suppose that $p_n \geq 0, q_n \geq 0$ and

$$\sum_{i=1}^{\frac{\gamma+1}{2}} \prod_{\tau=1}^{2i-1} p_{n-\kappa-\tau\mu} < 1 \qquad \textbf{(3)}$$

$$\limsup_{n\to\infty} \sum_{\gamma=n_1}^{n} \left[q_\gamma d_\gamma \left(1 - \sum_{i=1}^{\frac{\gamma+1}{2}} \prod_{\tau=1}^{2i-1} p_{n-\kappa-\tau\mu}\right)^{\zeta} - \frac{(\Delta d_\gamma)^2}{4 d_\gamma} \right] = \infty \qquad \textbf{(4)}$$

Where $d_l \in (\mathbb{R}, (0,\infty))$ then all solutions of **Equation (1)** oscillate**.**
Proof**:** Assume that Y_n is not oscillatory of **Equation**(**1**)
Suppose Y_n is eventually positive; there exists n_0 such that $Y_n > 0$, $Y_{n-\mu} > 0$, $Y_{n-\kappa} > 0$ for $n > n_0$, Let $F_n = Y_n + p_n Y_{n-\mu} > 0, F_n \geq 0$, $\Delta^2(Y_n + p_n Y_{n-\mu}) + p_n Y_{n-\mu} = 0$

$\Delta^2 Ƒ_{ɳ} + q_{ɳ} Υ_{ɳ-\,Ƙ} = 0,\ \Delta^2 Ƒ_{ɳ} = -q_{ɳ} Υ_{ɳ-\,Ƙ} \le 0$
Such that $q_{ɳ} \ge 0, Υ_{ɳ-\,Ƙ} > 0$, Either $\Delta Ƒ_{ɳ} \ge 0$ or $Ƒ_{ɳ} \le 0$
If $\Delta Ƒ_{ɳ} \le 0$, for $ɳ \ge ɳ_1 \ge ɳ_0$ leads to $z_{ɳ} \le 0$ which is a contradiction.
Hence $\Delta Ƒ_{ɳ} \ge 0$, and $\Delta^2 Ƒ_{ɳ} \le 0$
$\Delta Ƒ_{ɳ}$ non-increasing
$Ƒ_{ɳ} = Υ_{ɳ} + ƥ_{ɳ} Υ_{ɳ-\mu}$
$Υ_{ɳ} = Ƒ_{ɳ} - ƥ_{ɳ} Υ_{ɳ-\mu}$
$Υ_{ɳ} = Ƒ_{ɳ} - ƥ_{ɳ}[Ƒ_{ɳ-\mu} + ƥ_{ɳ-\mu} Υ_{ɳ-2\mu}]$
$Υ_{ɳ} = Ƒ_{ɳ} - ƥ_{ɳ}\left[Ƒ_{ɳ-\mu} - ƥ_{ɳ-\mu}(Ƒ_{ɳ-2\mu} - ƥ_{ɳ-2\mu} Υ_{ɳ-3\mu})\right]$
$= Ƒ_{ɳ} - ƥ_{ɳ} Ƒ + ƥ_{ɳ} ƥ_{ɳ-\mu} Ƒ_{ɳ-2\mu} - ƥ_{ɳ} ƥ_{ɳ-2\mu} Υ_{ɳ-3\mu}$
In general, if j is ≥ 3 , it can be obtained that: $Υ_{ɳ} = Ƒ_{ɳ} + \prod_{i=0}^{j} ƥ_{ɳ-i\mu} Υ_{ɳ-(i+1)\mu}$

$$Υ_{ɳ} = Ƒ_{ɳ} + \prod_{i=0}^{j} ƥ_{ɳ-i\mu} Υ_{ɳ-(i+1)\mu} + \sum_{i=0}^{\frac{j-1}{2}} \prod_{ɤ=0}^{2i-1} ƥ_{ɳ-ɤ\mu} Ƒ_{ɳ-2i\mu} - \sum_{i=1}^{\frac{j+1}{2}} \prod_{ɤ=1}^{2i-1} ƥ_{ɳ-ɤ\mu} Ƒ_{ɳ-(2i-1)\mu}.$$

$$Υ_{ɳ} \ge Ƒ_{ɳ} - \sum_{i=1}^{\frac{j+1}{2}} \prod_{ɤ=1}^{2i-1} ƥ_{ɳ-ɤ\mu} Ƒ_{ɳ-(2i-1)\mu}$$

For $ɳ \ge ɳ - (2i-1)\mu$

$$Υ_{ɳ} \ge \left(1 - \sum_{i=1}^{\frac{j+1}{2}} \prod_{ɤ=1}^{2i-1} ƥ_{ɳ-\,ɤ\mu}\right) Ƒ_{ɳ}$$

$$Υ_{ɳ-\,Ƙ} \ge \left(1 - \sum_{i=1}^{\frac{j+1}{2}} \prod_{ɤ=1}^{2i-1} ƥ_{ɳ-\,Ƙ-ɤ\mu}\right) Ƒ_{ɳ-\,Ƙ}$$

Define

$$\Theta_{ɳ} = \frac{ɗ_{ɳ} \Delta Ƒ_{ɳ}}{Ƒ^{\alpha}_{ɳ-\,Ƙ}}$$

It is obvious that $\Theta_{ɳ} > 0$, then

$$\Delta^2 Ƒ_{ɳ} = \frac{Ƒ^{\alpha}_{ɳ-\,Ƙ}}{ɗ_{ɳ}} \Delta\Theta_{ɳ} + \Theta_{ɳ+1} \Delta\left(\frac{Ƒ^{\alpha}_{ɳ-\,Ƙ}}{ɗ_{ɳ}}\right)$$

$$= \frac{Ƒ^{\alpha}_{ɳ-\,Ƙ}}{ɗ_{ɳ}} \Delta\Theta_{ɳ} + \Theta_{ɳ+1}\left(\frac{ɗ_{ɳ}\, \Delta Ƒ^{\alpha}_{ɳ-\,Ƙ} - Ƒ^{\alpha}_{ɳ-v} \Delta ɗ_{ɳ}}{ɗ_{ɳ} ɗ_{ɳ+1}}\right)$$

Substituting

$$\frac{Ƒ^{\alpha}_{ɳ-\,Ƙ}}{ɗ_{ɳ}} \Delta\Theta_{ɳ} + \Theta_{ɳ+1}\left(\frac{ɗ_{ɳ} \Delta Ƒ^{\alpha}_{ɳ-\,Ƙ} - Ƒ^{\alpha}_{ɳ-\,Ƙ} \Delta ɗ_{ɳ}}{ɗ_{ɳ} ɗ_{ɳ+1}}\right) + q_{n}\left(1 - \sum_{i=1}^{\frac{j+1}{2}} \prod_{ɤ=1}^{2i-1} ƥ_{ɳ-\,Ƙ-ɤ\mu}\right)^{\zeta} Ƒ^{\alpha}_{ɳ-k} \le 0$$

$$\Delta\Theta_{ɳ} + \frac{ɗ_{ɳ}}{Ƒ^{\alpha}_{ɳ-\,Ƙ}} \Theta_{ɳ+1}\left(\frac{ɗ_{ɳ} Ƒ^{\alpha}_{ɳ-\,Ƙ} - Ƒ^{\alpha}_{ɳ-\,Ƙ} \Delta ɗ_{ɳ}}{ɗ_{ɳ} ɗ_{ɳ+1}}\right) + \frac{ƥ_{ɳ}}{Ƒ^{\alpha}_{ɳ-\,Ƙ}} q_{n}\left(1 - \sum_{i=1}^{\frac{j+1}{2}} \prod_{ɤ=1}^{2i-1} ƥ_{ɳ-\,Ƙ-ɤ\mu}\right)^{\zeta} Ƒ^{\alpha}_{ɳ-\,Ƙ} \le 0$$

$$\Delta\Theta_{n} \leq -\frac{d_{n}}{F^{\alpha}_{n-\kappa}}\Theta_{n+1}\left(\frac{d_{n}\Delta F^{\alpha}_{n-\kappa} - F^{\alpha}_{n-\kappa}\Delta d_{n}}{d_{n}d_{n+1}}\right) - q_{n}d_{n}\left(1-\sum_{i=1}^{\frac{j+1}{2}}\prod_{\gamma=1}^{2i-1}p_{n-\kappa-\gamma\mu}\right)^{\zeta}$$

$\Delta F^{2}_{n} \leq 0$, then ΔF_{n} decreasing
$\Delta F^{\alpha}_{n-\kappa+1} \leq \Delta F^{\alpha}_{n-\kappa}$ and $F^{\alpha}_{n-\kappa+1} > F^{\alpha}_{n-\kappa}$

$$\Delta\Theta_{n} \leq -d_{n}\Theta_{n+1}\frac{\Delta F^{\alpha}_{n-\kappa+1}}{d_{n}F^{\alpha}_{n-\kappa+1}} + \frac{\Delta d_{n}}{d_{n+1}}\Theta_{n+1} - q_{n}d_{n}\left(1-\sum_{i=1}^{\frac{j+1}{2}}\prod_{\gamma=1}^{2i-1}p_{n-\kappa-\gamma\mu}\right)^{\zeta}$$

$$\leq -\frac{d_{n}}{d^{2}{}_{n+1}}\left(\Theta_{n+1} - \frac{d_{n+1}\Delta d_{n}}{2d_{n}}\right)^{2} + \frac{(\Delta d_{n})^{2}}{4d_{n}} - d_{n}p_{n}\left(1-\sum_{i=1}^{\frac{j+1}{2}}\prod_{\gamma=1}^{2i-1}p_{n-\kappa-\gamma\mu}\right)^{\zeta}$$

$$\Delta\Theta_{n} \leq -\left[q_{n}d_{n}\left(1-\sum_{i=1}^{\frac{j+1}{2}}\prod_{\gamma=1}^{2i-1}p_{n-\kappa-\gamma\mu}\right)^{\zeta} - \frac{(\Delta d_{n})^{2}}{4d_{n}}\right]$$

By performing sum of n_1 to n to last inequality:

$$\sum_{l=n1}^{n}\Delta\Theta_{y} \leq -\sum_{y=n1}^{n}\left[d_{y}q_{y}\left(1-\sum_{i=1}^{\frac{y+1}{2}}\prod_{\gamma=1}^{2i-1}p_{n-\kappa-\gamma\mu}\right)^{\zeta} - \frac{(\Delta d_{y})^{2}}{4d_{y}}\right]$$

$$\Theta_{n+1} - \Theta_{n1} \leq -\sum_{y=n1}^{n}\left[q_{y}d_{y}\left(1-\sum_{i=1}^{\frac{y+1}{2}}\prod_{\gamma=1}^{2i-1}p_{n-\kappa-\gamma\mu}\right)^{\zeta} - \frac{(\Delta d_{y})^{2}}{4d_{y}}\right]$$

As n goes to ∞ with (4), the previous inequality reduces that $\lim_{n\to\infty}\Theta_{n} = -\infty$, a contradiction.

3.2. Theorem: Suppose that $0 \leq p_2 \leq p_2 < 1$, $q_{n} \leq q < 0$. Then all non-oscillatory solutions of **Equation (1)** approaches to ∞ as $n \to \infty$ or approaches to zero.

Proof: Assuming that the solution Y_{n} is eventually positive, it's reducing that there is a positive integer $n_0 > 0$, $\ni Y_{n} > 0, Y_{n-\mu} > 0, Y_{n-\kappa} > 0$, $n \geq n_0$, in the same strategy of (3.1. theorem):

The sequences $F_{n}, \Delta F_{n}$ are monotone, so two cases for ΔF_{n} must be considered:

$\Delta F_{n} > 0$, $n \geq n_1 \geq n_0$

$\Delta F_{n} < 0$, $n \geq n_1 \geq n_0$

Case 1. $\Delta^{2}F_{n} \geq 0, \Delta F_{n} > 0, F_{n} > 0$

With use (2.2 lemma), reducing that $\lim_{n\to\infty}F_{n} = \infty$

$Y_{n} = F_{n} - p_{n}Y_{n-\mu} \geq F_{n} - p_{2}Y_{n-\mu} \geq F_{n} - p_{2}F_{n-\mu} \geq (1-p_{2})F_{n}$.

as n goes to ∞ then, bygone inequality gets Y_{n} goes to ∞.

Case 2. If $\Delta^{2}F_{n} \geq 0, \Delta F_{n} < 0, F_{n} > 0$, $\ni n \geq n_1 \geq n_0$. Reducing F_{n} is decreasing the positive sequence; it is bounded with exist limit. Hence, Y_{n} is also bounded because of $F_{n} \geq Y_{n}$. Now, $\lim_{n\to\infty}\inf Y_{n} = 0$, By contradiction, let $\lim_{n\to\infty}\inf Y_{n} = a > 0$ as there exists n_2 such that $F_{n} \geq Y_{n} \geq a, n \geq n_2 \geq n_1$, So, $\Delta^{2}F_{n} = -q_{n}(Y^{\alpha}_{n-\kappa}) \geq -q(Y^{\alpha}_{n-\kappa})$

By performing sum of n_2 to $n-1$ to last inequality:

$$\Delta F_{n} - \Delta F_{n_2} \geq -\sum_{i=n_2}^{n-1} q(Y_{i-\kappa})$$

$$-\Delta F_{n_2} \geq -\sum_{i=n_2}^{n-1} q a^{\alpha}(Y_{i-\kappa})$$

As n goes to ∞ with the previous inequality reduces to a contradiction; thus, $\lim_{n\to\infty} \inf Y_{n} = 0$

With use (2.3 lemma), reducing that $\lim_{n\to\infty} \inf F_{n} = 0$.

3.3. Application Examples:

An application example was formulated to support the obtained results.

3.3.1.Example: Study the Emden-Fowler dynamic equation:

$$\Delta^2\left[Y_{n} + \tfrac{1}{2}Y_{n-2}\right] + 5Y_{n-3} = 0, \quad \textbf{(5)}$$

When $Y_{n} = (-1)^{n}$, n is positive integer number, it satisfies the difference equations as follows:

$$\Delta\left[\Delta\left(Y_{n} + \tfrac{1}{2}Y_{n-2}\right)\right] + 5Y_{n-3} = \Delta\left[Y_{n+1} - Y_{n} + \tfrac{1}{2}Y_{n-1} - \tfrac{1}{2}Y_{n-2}\right] + 5Y_{n-3}$$

$$= Y_{n+2} - 2Y_{n+1} + \frac{3}{2}Y_{n} - Y_{n-1} + \frac{1}{2}Y_{n-2.} + 5Y_{n-3}$$

Then

$3(-1)^{n} + 2(-1)^{n} - 5(-1)^{n} = 0$. To satisfy all conditions of theorem 3.1:

Since $p_{n} = \frac{1}{2}, q_{n} = 6, d_{n} = 3n, \mu = 1,\ \kappa = 3, i = 3, \zeta = 1$ then

$$\sum_{i=1}^{2}\prod_{\gamma=1}^{2i-1} p_{n-\kappa-\gamma\mu} = \frac{1}{4} < 1$$

And

$$\limsup_{n\to\infty}\sum_{y=n_1}^{n}\left[q_l d_l\left(1-\sum_{i=1}^{\frac{y+1}{2}}\prod_{\gamma=1}^{2i-1} p_{n-\kappa-\gamma\mu}\right)^{\zeta} - \frac{(\Delta d_y)^2}{4d_y}\right]$$

$$= \limsup_{n\to\infty}\sum_{y=n_1}^{n}\left[18y\left(1-\frac{1}{4}\right) - \frac{1}{12y}\right]$$

$$= \lim_{n\to\infty} sup \sum_{n_1}^{n}\left[\frac{27y}{2} - \frac{1}{12y}\right] = \infty$$

Then by (3.1. theorem), all solutions oscillate.

3.3.2. Example: Study the Emden-Fowler dynamic equation:

$$\Delta^2\left[Y_{n} + \tfrac{1}{3}Y_{n-2}\right] + \tfrac{16}{3}Y_{n-3}^{\frac{1}{3}} = 0, n \text{ is a positive integer number} \quad \textbf{(6)}$$

Let $Y_{n} = (-1)^{n}$ it satisfies the difference equations as follows:

$$\Delta\left[\Delta Y_{n} + \tfrac{1}{3}\Delta Y_{n-2}\right] + \tfrac{16}{3}Y_{n-3}^{\frac{1}{3}} = \Delta\left[Y_{n+1} - Y_{n} + \tfrac{1}{3}Y_{n-1} - \tfrac{1}{3}Y_{n-2}\right] + \tfrac{16}{3}Y_{n-3}^{\frac{1}{3}}$$

$$= Y_{n+2} - 2Y_{n+1} + \frac{4}{3}Y_{n} - \frac{2}{3}Y_{n-1} + \frac{1}{3}Y_{n-2} + \frac{16}{3}Y_{n-3}^{\frac{1}{3}}$$

$$= (-1)^{n+2} - 2(-1)^{n+1} + \frac{4}{3}(-1)^{n} - \frac{2}{3}(-1)^{n-1} + \frac{1}{3}(-1)^{n-2} + \frac{16}{3}(-1)^{\frac{n}{3}-1}$$

$$= 3(-1)^{n} + \frac{7}{3}(-1)^{n} - \frac{16}{3}(-1)^{n} = 0$$

Since $\text{þ}_{n} = \frac{1}{3}, q_{n} = \frac{16}{3}, \text{ɗ}_{n} = n + 1, \mu = 1, \; K = 3, i = 3, \zeta = \frac{1}{3}$ then

$$\sum_{i=1}^{2}\prod_{\gamma=1}^{2i-1} \text{þ}_{n - K - \gamma\mu} = \frac{2}{27} < 1$$

$$\limsup_{n\to\infty} \sum_{\text{ɣ}=n_1}^{n}\left[q_l \text{ɗ}_l \left(1 - \sum_{i=1}^{\frac{\text{ɣ}+1}{2}}\prod_{\gamma=1}^{2i-1} \text{þ}_{n - K - \gamma\mu}\right)^{\zeta} - \frac{(\Delta \text{ɗ}_{\text{ɣ}})^2}{4\text{ɗ}_{\text{ɣ}}}\right] = \infty$$

$$\lim_{n\to\infty} sup \sum_{i=n_1}^{n}\left[\frac{16(\text{ɣ}+1)}{3}\left(1 - \frac{2}{27}\right)^{\frac{1}{3}} - \frac{1}{4(\text{ɣ}+1)}\right] = \infty$$

Also, we get:

$$\lim_{n\to\infty} sup \sum_{i=n_1}^{n}\left[\frac{16\sqrt[3]{25}}{9}(\text{ɣ}+1) - \frac{1}{4(\text{ɣ}+1)}\right] = \infty$$

Then by (3.1. theorem), all solutions oscillate.

3.3.3.Example: Study the Emden-Fowler dynamic equation:

$$\Delta^2\left[Y_{n} + \frac{1}{3}Y_{n-3}\right] - \frac{2}{3(n-1)}Y_{n-1} = 0 \qquad \textbf{(7)}$$

where $\text{þ}_{n} = \frac{1}{3} < 1, q_{n} = -\frac{2}{3(n-1)}, n > 1$

$$\Delta^2\left[Y_{n} + \frac{1}{3}Y_{n-3}\right] - \frac{2}{3(n-1)}Y_{n-1} = \Delta\left[Y_{n+1} - Y_{n} + \frac{1}{3}Y_{n-2} + \frac{1}{3}Y_{n-3}\right] - \frac{2}{3(n-1)}Y_{n-1}$$

$$= \left[Y_{n+2} - 2Y_{n+1} + Y_{n} + \frac{1}{3}Y_{n-1} - \frac{1}{3}Y_{n-3}\right] - \frac{2}{3(n-1)}Y_{n-1}$$

$$= \left[\frac{n+2}{3} - \frac{2(n+1)}{3} + \frac{n}{3} + \frac{n-1}{9} - \frac{n-3}{9}\right] - \frac{2(n-1)}{9(n-1)}$$

$$= \frac{3n + 6 - 6n - 6 + 3n + n - 1 - n + 3}{9} - \frac{2}{9} = 0$$

Then by (3.2. theorem), every solution tends to zero or infinity, such as $Y_{n} = \frac{n}{3}$.

4. Discussion

This study focuses on the oscillation and asymptotic behavior of all solutions to Emden–Fowler difference equations. The results include the establishment of new criteria and conditions for both oscillation and non-oscillation, building on and extending existing results in the literature[23,24]. From the oscillatory perspective, the findings indicate that the behavior of solutions is significantly influenced by the interaction between linear and nonlinear terms, as well as their coefficient functions[25,26]. Regarding asymptotic properties, the results demonstrate that solutions exhibit a variety of long-term behaviors depending on the coefficients. Under suitable conditions, solutions may converge to zero or approach finite constants. The applicability of these criteria is further illustrated through representative examples.

5. Conclusion

In this paper, the oscillatory and asymptotic properties of Emden–Fowler difference equations are investigated. New conditions are established for the given functions in Equation (1). The proposed conditions for determining oscillatory and asymptotic behavior are effective, flexible, and easy to apply. In comparison with existing results, these conditions are less restrictive and applicable to a broad class of Emden–Fowler difference equations on time scales. Illustrative

examples are provided to demonstrate the significance of the main results in comparison with previously published work. Several known theorems in the literature are thereby improved.
Furthermore, the results highlight that the time scale calculus provides a robust framework for analyzing various forms of such equations. This unified approach allows results from differential and difference equations to be extended to more general settings. It not only simplifies theoretical analysis but also broadens applicability to real-world problems, where processes may evolve either continuously or discretely. The oscillation and convergence properties of Emden–Fowler dynamic equations depend strongly on the coefficients as well as on the order of the equation. As future work, it would be of interest to extend the present results to higher-order difference equations with forcing terms.

Acknowledgment

Our acknowledgments are to all authors whose results we have improved and to reviewers for their remarks.

Conflict of Interest

The authors declare that they have no conflicts of interest.

Funding

No funding.

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