Some Coincidence and Common Fixed Point Theorems for Two Self Mappings under Generalized Contractive Condition in Cone-b- Metric Space

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Abstract

In this paper, we prove some coincidence and common fixed point theorems for a pair of discontinuous weakly compatible self mappings satisfying generalized contractive condition in the setting of Cone-b- metric space under assumption that the Cone which is used is non-normal. Our results are generalizations of some recent results.

Key Words: Coincidence and common fixed point, pair of weakly compatible mappings, generalized contractive self mapping, Cone-b- metric space, normal Cone, non-normal Cone.

Introduction

Metric fixed point theory is a branch of fixed point theory which finds its primary application in functional analysis. It is a sub-branch of the functional analytic theory in which geometric conditions on the mapping and / or underlying space play a crucial role. Although it has a purely metric facet, it is also a major branch of nonlinear functional analysis with close ties to Banach space geometry, [1]. Historically; the basic idea of metric fixed point principle firstly appeared in explicit from Banach's thesis 1922 [2,p.5], where it was used to establish the existence of solution to an integral equation. This principle Banach contraction mapping is remarkable in its simplicity contraction; it is perhaps the most widely applied fixed point theorem in all of analysis. This is because the contractive condition on the mapping is simple and easy to test because:

(i) IT requires only complete metric space for its setting.

(ii) IT provides a contractive algorithm (iterative method).

(iii) IT finds almost conociale applications in the theory of differential and integral equations specially the existence solution, uniqueness solution.

All these properties motivate authors to study this principle and there appeared many types of contraction mapping on metric space.

Recently, Bakhtin [3] introduced b-metric space as a generalization of metric spaces. He proved the Contraction mapping principle in b-metric spaces that generalized the famous Banach Contraction principle in metric spaces. Since then, several papers have dealt with fixed point theory or the variation principle for single-valued and multi-valued operators in b-metric spaces (as shown in [4]and [5]). in [6] Haung and Zhang introduced Cone metric spaces as a generalization of metric spaces by replacing the set of real numbers by an ordered Banach space and they proved some fixed point theorems for contractive mappings by using the normality of a Cone in results which expanded certain results of fixed points in metric spaces, and other authors who worked in the same way like [7] and [8]. In [9], Hussain and Shah introduced Cone b-metric spaces as a generalization of b-metric spaces and Cone metric spaces and they established some topological properties in such space and improved some recent results about KKM mappings in the setting of a Cone b-metric space, as well as in [10] they generalized the results of [9] and obtained some fixed point theorems of contractive mappings without the assumption of normality of the Cone. In this paper, we generalized the results of [9] and [10] and prove some coincidence and common fixed point theorems for a pair of discontinuous weakly compatible self mappings satisfying generalized contractive condition by using a certain vector valued altering function satisfying some properties in the setting of Cone-b- metric space where the normality of the Cone is omitted, we shall call this altering function by Cone-b-altering function.

Preliminaries

Consistent with Haung and Zhang [6], the following definitions:

Let E be a normed space and P be a subset of E, P is called a Cone if:

(i) P is closed, non empty and $P \neq \{0\}$.

(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b.

(iii) $P \cap (-P) = \{0\}$.

Given a Cone $P \subset E$, we define a partial ordering "$\leq$" with respect to P by $x \leq y$ if and only if $y - x \in P$, we write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ stand for $y - x \in \text{int}(P)$, where int(P) is the interior of P.

The Cone P is called normal if there is a number k > 0 such that for all x, y ∈E, 0 ≤ x ≤ y implies ‖x‖ ≤ k‖y‖, the least positive number satisfying the above inequality is called the normal constant of P.

Example (2.1): [7]
Let E =C_R([0,1]) with supremum norm and P ={ f∈ E :f ≥ 0 }where || f|| =sup {|f(x_i)| , x_i ∈[0,1] } for all f, g ∈ P , put f(x) =x , g(x) =2x , then 0≤f ≤ g , ||f|| =1 , ||g|| =2 . So ||f|| ≤||g|| and K=1 . Therefore P is normal cone with normal constant K=1.

Remark (2.2):[7]
There are cones are not normal ,the following example show that :

Example (2.3):[7]
Let E =C_R^2([0,1]) with the norm ||f|| =$||f||_\infty$ +$||f'||_\infty$ and consider the cone P ={ f∈ E :f ≥ 0 } , where $||f||_\infty$ =max {|f(x_1)|, |f(x_2)|,…. |f(x_n)|, x_i ∈[0,1] ∀I =1,2,…..,n }
$||f'||_\infty$ =max {|f′(x_1), |f′(x_2),…., |f′(x_n)|, x_i ∈[0,1] ∀I =1,2,…..,n }
For each k≥1 ,put f(x)=x and g(x) =x^{2k} . Then 0≤g≤f , ||f|| =2 and ||g|| =2k+1, since k||f||<||g||, k is not a normal constant of P. Therefore , P is non –normal cone .
In the following we always suppose that E is a normed space , P is a cone in E with int(p)≠∅ and ≤ is a partial ordering with respect to P .

Definition 2.4: [6]

Let X be non-empty set, a mapping d: X×X ⟶ E is called a Cone metric space on X if the following conditions are satisfied:
(i) 0 ≤ d(x,y) for all x, y ∈ X with x ≠ y and d(x,y) = 0 if and only if x = y.
(ii) d(x,y) = d(y,x) for all x, y ∈ X.
(iii) d(x,y) ≤ d(x,z) + d(z,y) for all x, y, z ∈ X.
Then the ordered pair (X,d) is called a Cone metric space.

Example 2.5 :[6]
Let E =R^2 with usual norm on R^2 defined by ||x|| =max{|x_1|,|x_2 |} for all x∈R^2 , x=(x_1,x_2), x_i∈R,i=1,2, P={(x,y) ∈ E: x,y≥0} ⊂R^2 , X=R and d:X×X→E such that :d(x,y)=(|x-y| ,α|x-y|), where α≥0 is a constsnt .Then (X,d) is a cone metric space .

Definition 2.6: [9]

Let X be a non empty set and S ≥ 1 be a given real number. A mapping d: X×X ⟶ E is said to be Cone b-metric if and only if, for all x, y, z ∈ X, the following conditions are satisfied:
(i) 0 < d(x,y) with x ≠ y and d(x,y) = 0 if and only if x = y.
(ii) d(x,y) = d(y,x).
(iii) d(x,y) ≤ S[d(x,z) + d(z,y)].
The pair (X,d) is called a Cone-b-metric space.

Example 2.7:[10]
Let X={1,2,3,4}, E=R^2, P={(x,y)∈ E :x≥0, y≥0 }. Define d:X×X→E by

$$d(x,y) = \begin{cases} \left(|x-y|^{-1},\ |x-y|^{-1}\right) & \text{if } x \neq y \\ \theta & \text{if } x=y \end{cases}$$

Then (X,d) is a cone b-metric space with the coefficient $S = \frac{6}{5}$.

Definition (2.8): [9]

Let (X,d) be a Cone b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converge to x whenever for every $c \in E$ with $0 \ll c$, there is a natural number N such that $d(x_n,x) \ll c$ for all n > N. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$.

(ii) $\{x_n\}$ is Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number N such that $d(x_n,x_m) \ll c$ for all n, m > N.

(iii) (X,d) is a complete Cone-b-metric space if every Cauchy sequence is convergent.

Definition (2.9): [11]

If Y be any partially ordered set with relation "≤" and $f : Y \longrightarrow Y$, we say that f is non-deceasing if, $x, y \in Y$, $x \le y \Rightarrow f(x) \le f(y)$.

Definition (2.10): [11]

A function f: $P \longrightarrow P$ is called subadditive if for all $x, y \in P$, $f(x + y) \le f(x) + f(y)$.

Seong-Hoon Cho [12] defined the ≪-increasing function by following:

A function $F:P \longrightarrow P$ is called ≪-increasing if for each $x, y \in P$, $x \ll y$ if and only if $F(x) \ll F(y)$.

In the following we shall introduce Cone-b-altering function.

Definition (2.11):

Let (X,d) be a Cone-b-metric space, let $F:P \longrightarrow P$ be a vector valued function, F is called a Cone-b-altering function if:

(i) F is non-decreasing, subadditive, ≪-increasing and surjective.

(ii) If,for $\{t_n\} \subset p$, $\lim_{n\to\infty} F(t_n) = 0 \leftrightarrow \lim_{n\to\infty} t_n = 0$

(iii) $F(\alpha^k t) = \alpha^k F(t)$ for $\alpha \ge 1$, k=1,2,…….

Example (2.12):

Let $F(t) = t$ for all $t \in P$ then F is Cone-b-altering function.

The following lemmas which are necessary through our work in this sequel are often used in Cone metric spaces in which the Cone need not be normal.

Lemma (2.13): [8]

Let P be a Cone and $\{a_n\}$ be a sequence in E. If $c \in int(P)$ and $0 \le a_n \to 0$ (as $n \to \infty$), then there exists N such that for all n > N, we have $a_n \ll c$.

Lemma (2.14): [8]

Let $x, y,z \in E$, if $x \ll y$ and $y \ll z$ then $x \ll z$.

Lemma (2.15): [9]

Let P be a Cone and $0 \le u \ll c$ for each $c \in int(P)$, then $u = 0$.

Lemma (2.16): [13]

Let P be a Cone. If $u \in P$ and $u \le ku$ for some $0 \le k < 1$, then $u = 0$.

The following definitions and proposition are necessary in this sequel.

Definition (2.17): [11]

Let X be any non-empty set, $f, g:X \longrightarrow X$ be mappings, a point $w \in X$ is called point of coincidence of f and g if there is $x \in X$ such that $fx = gx = w$.

Definition (2.18): [11]

Let X be any non-empty set, $f, g:X \longrightarrow X$ be mappings, the pair (f, g) is called weakly compatible if $x \in X$, $fx = gx \Rightarrow fgx = gfx$.

Proposition (2.19): [11]

Let X be any non-empty set and $f, g:X \longrightarrow X$ be mappings. If (f,g) is weakly compatible pair and have a unique point of coincidence then it is unique common fixed point of f and g.

Main Result

Theorem (3.1):

Let (X,d) be a Cone metric space with the coefficient $S \geq 1$, suppose the mappings $f, g:X \longrightarrow X$ satisfying for all $x, y \in X$:

$$F[d(fx,fy)] \leq a_1F[d(gx,gy)] + a_2F[d(fx,gx)] + a_3F[d(fy,gy)]+ a_4F[d(fx,gy)] + a_5F[d(fy,gx)] \quad \ldots(3.1.1)$$

where this constant $a_i \in [0,1)$ and $a_1 + a_2 + a_3 + S(a_4 + a_5) < 1$, $i = 1,2,3,4,5$ and F be Cone-b-altering function. If $f(X) \subset g(X)$ and $f(X)$ is complete, then f and g have a unique point of coincidence. Furthermore if the pair (f,g) is weakly compatible pair then f, g have a unique common fixed point.

Proof:

Let $x_0 \in X$ be arbitrary point in X. Since $f(X) \subset g(X)$, we can choose a point x_1 in X such that $f x_0 = g x_1$, if we continue in same way we can choose x_{n+1} in X for x_n in X such that $gx_{n+1} = fx_n$ for all $n \geq 0$.

If $x = x_{n+1}$, $y = x_n$ in (3.1.1), we have:

$$F[d(fx_{n+1},fx_n)] \leq a_1F[d(gx_{n+1},gx_n)] + a_2F[d(fx_{n+1},gx_{n+1})] + a_3F[d(fx_n,gx_n)]+a_4F[d(fx_{n+1},gx_n)]+ a_5F[d(fx_n,gx_{n+1})]$$

$$F[d(fx_{n+1}, fx_n)] \leq a_1F[d(fx_n, fx_{n-1})] + a_2F[d(fx_{n+1}, fx_n)] + a_3F[d(fx_n, fx_{n-1})]+ a_4F[d(fx_{n+1}, fx_{n-1})] + a_5F[d(fx_n, fx_n)]$$

$$\leq a_1F[d(fx_n, fx_{n-1})] + a_2F[d(fx_{n+1}, fx_n)] + a_3F[d(fx_n, fx_{n-1})]+ Sa_4F[d(fx_{n+1}, fx_n)] + Sa_4F[d(fx_n, fx_{n-1})]$$

$$(1 - a_2 - a_4) F[d(fx_{n+1}, fx_n)] \leq (a_1 + a_3 + Sa_4) F[d(fx_n, fx_{n-1})] \quad ...(3.1.2)$$

Using symmetry of (3.1.2) in x, y we have:

$$(1 - a_3 - Sa_5) F[d(fx_{n+1}, fx_n)] \leq (a_1 + a_2 + Sa_5) F[d(fx_n, fx_{n-1})] \quad ...(3.1.3)$$

Now combine (3.1.2) and (3.1.3) we have:

$$F[d(fx_{n+1}, fx_n)] \leq \frac{2a_1+a_2+a_3+S(a_4+a_5)}{2-a_2-a_3-S(a_4+a_5)} F[d(fx_n, fx_{n-1})]$$

Put $\lambda = \dfrac{2a_1+a_2+a_3+S(a_4+a_5)}{2-a_2-a_3-S(a_4+a_5)}$,

We must prove that $\lambda < 1$.

Since $a_1 + a_2 + a_3 + S(a_4 + a_5) < 1$

$\Rightarrow$ $a_2 + a_3 + S(a_4 + a_5) < 1 - a_1$
$\Rightarrow$ $-a_2 - a_3 - S(a_4 + a_5) > a_1 - 1$
$\Rightarrow$ $2 - a_2 - a_3 - S(a_4 + a_5) > a_1 + 1$

$$\Rightarrow \frac{1}{2-a_2-a_3-S(a_4+a_5)} < \frac{1}{a_1+1}$$

$$\Rightarrow \frac{2a_1+a_2+a_3+S(a_4+a_5)}{2-a_2-a_3-S(a_4+a_5)} < \frac{2a_1+a_2+a_3+S(a_4+a_5)}{a_1+1} < 1$$

Therefore $0 \le \lambda < 1$, so we have
$F[d(fx_{n+1}, fx_n)] \le \lambda F[d(fx_n, fx_{n-1})] \le \ldots \le \lambda^n F[d(fx_1, fx_0)]$
Now, for any $m \ge 1$, $p \ge 1$, it follows that

$$\begin{aligned} d(fx_{m+p}, fx_m) &\le S[d(fx_{m+p}, fx_{m+p-1}) + d(fx_{m+p-1}, fx_m)] \\ &= Sd(fx_{m+p}, fx_{m+p-1}) + Sd(fx_{m+p-1}, fx_m) \\ &\le S\, d(fx_{m+p}, fx_{m+p-1})] + s^2[d(fx_{m+p-1}, fx_{m+p-2}) + d(fx_{m+p-2}, fx_m)] \\ &= Sd(fx_{m+p}, fx_{m+p-1}) + s^2 d(fx_{m+p-1}, fx_{m+p-2})] + s^2 d(fx_{m+p-2}, fx_m) \\ &\le Sd(fx_{m+p}, fx_{m+p-1}) + s^2 d(fx_{m+p-1}, fx_{m+p-2})] + s^2 d(fx_{m+p-2}, fx_m) + s^3 d(fx_{m+p-2}, fx_{m+p-3}) + \ldots + s^{p-1} d(fx_{m+2}, fx_{m+1}) + s^{p-1} d(fx_{m+1}, fx_m). \end{aligned}$$

But by (i) of definition (2.11)of F; F is non –decreasing and sub additive function we have :
$F[d(fx_{m+p}, fx_m)]] \le F[Sd(fx_{m+p}, fx_{m+p-1})] + F[s^2 d(fx_{m+p-1}, fx_{m+p-2})] + F[s^3 d(fx_{m+p-2}, fx_{m+p-3})] + \ldots + F[s^{p-1} d(fx_{m+2}, fx_{m+1})] + F[s^{p-1} d(fx_{m+1}, fx_m)]$
Also by (iii) of above definition we have :

$F[d(fx_{m+p}, fx_m)]] \le SF[d(fx_{m+p}, fx_{m+p-1})] + s^2 F[d(fx_{m+p-1}, fx_{m+p-2})] + s^3 F[d(fx_{m+p-2}, fx_{m+p-3})] + \ldots + s^{p-1} F[d(fx_{m+2}, fx_{m+1})] + s^{p-1} F[d(fx_{m+1}, fx_m)]$
$\le s\lambda^{m+p-1} F[d\{fx_1,fx_0)] + s^2 \lambda^{m+p-2} F[d\{fx_1,fx_0)] + s^3 \lambda^{m+p-3} F[d\{fx_1,fx_0)] + \ldots + s^{p-1} \lambda^{m+1} F[d\{fx_1,fx_0)] + s^{p-1} \lambda^m F[d\{fx_1,fx_0)]$.

$$= \frac{s\lambda^{m+p}[(s\lambda^{-1})^{p-1} - 1]}{s-\lambda} F[d\{fx_1,fx_0)] + s^{p-1} \lambda^m F[d\{fx_1,fx_0)]$$

$$\le \frac{s^p \lambda^{m+1}}{s-\lambda} F[d\{fx_1,fx_0)] + s^{p-1} \lambda^m F[d\{fx_1,fx_0)] \to 0 \text{ as } m \to \infty$$

Hence, $\lim_{n,m\to\infty} d(fx_{m+p}, fx_m) = 0$ by (ii) of definition (2.11) of F. So by lemma (2.13), there exists $k \in N$ such that $d(fx_{m+p}, fx_m) \ll c$ for each $c \in int(P)$ and for all $m > k$.If $n=m+p$,so for all $m,n > k$, $\{fx_n\}$ is a Cauchy sequence in $f(X)$, but $f(X)$ is complete, so the sequence $\{fx_n\}$ must be convergent in $f(X)$, so there exists $u \in f(X)$ and $fx_n \to u$.
Now, since $u \in f(X) \subset g(X)$, let $u = g(v)$ for some $v \in X$. We show that $gv = fv$.
$d(fv,u) \le S[d(fv,fx_n)] + d(fx_n,u)]$
$d(fv,u) \le Sd(fv,fx_n) + Sd(fx_n,u)$
So by properties of F we have:
$F[d(fv,u)] \le SF[d(fv,fx_n)] + SF[d(fx_n,u)]$
By (3.1.1), we have:

$$\begin{aligned} F[d(fv,u)] &\le S[a_1F[d(gv,gx_n)] + a_2F[d(fv,gv)] + a_3F[d(fx_n,gx_n)] + a_4F[d(fv,gx_n)] + a_5F[d(fx_n,gv)]] + SF[d(fx_n,u)] \\ &\le S[a_1F[d(u, fx_{n-1})] + a_2F[d(fv,u)] + a_3F[d(fx_n, fx_{n-1})] + a_4F[d(fv, fx_{n-1})] + a_5F[d(fx_n,u)]\,] + SF[d(fx_n,u)] \\ &= Sa_1F[d(u, fx_{n-1})] + Sa_2F[d(fv,u)] + Sa_3F[d(fx_n, fx_{n-1})] + Sa_4F[d(fv, fx_{n-1})] + Sa_5F[d(fx_n,u)] + SF[d(fx_n,u)] \end{aligned}$$

$\leq Sa_1F[d(u, fx_{n-1})] + Sa_2F[d(fv,u)] + S^2a_3F[d(fx_n,u)] + S^2a_3F[d(u, fx_{n-1})] + S^2a_4F[d(fv,u)] + S^2a_4F[d(u, fx_{n-1})] + Sa_5F[d(fx_n,u)] + SF[d(fx_n,u)]$

That implies:

$$(1 - Sa_2 - S^2a_4) \leq \frac{Sa_1 + S^2a_3 + S^2a_4}{1 - Sa_2 - S^2a_4} F[d(u, fx_{n-1})] + \frac{S^2a_3 + Sa_5 + S}{1 - Sa_2 - S^2a_4} F[d(fx_n,u)]$$

Now, let $c \in \text{int}(P)$ be given. We can choose $n_0 \in N$ such that

$F[d(fx_{n-1},u)] \ll F^{-1}\left(\frac{1 - Sa_2 - S^2a_4}{Sa_1 + S^2a_3 + S^2a_4} \cdot \frac{c}{2}\right)$ and

$F[d(fx_n,u)] \ll F^{-1}\left(\frac{1 - Sa_2 - S^2a_4}{S^2a_3 + Sa_5 + S} \cdot \frac{c}{2}\right)$ for all $n > n_0$.

So by (i) of definition (2.11) of F; F is ≪-increasing and surjective, we have:

$F[d(fx_{n-1},u)] \ll \left(\frac{1 - Sa_2 - S^2a_4}{Sa_1 + S^2a_3 + S^2a_4} \cdot \frac{c}{2}\right)$ and

$F[d(fx_n,u)] \ll F^{-1}\left(\frac{1 - Sa_2 - S^2a_4}{S^2a_3 + Sa_5 + S} \cdot \frac{c}{2}\right)$

So that implies; $F[d(fv,u)] \ll c$, thus by lemma (2.15) we have $F[d(fv,u)] = 0$ and so by (ii) of definition (2.11) of F; we obtain that $d(fv,u) = 0$ and so by $fv = u = g(v)$, thus u is a point of coincidence of *f* and *g*.

To prove u is unique, suppose u′ is another point of coincidence then there is $v' \in X$ such that $u' = f v' = g v$, so by (3.1.1) we have:

$F[d(fv, fv')] \leq a_1F[d(gv,gv')] + a_2F[d(fv,gv)] + a_3F[d(fv',gv')] + a_4F[d(fv,gv')] + a_5F[d(fv',gv)]$

$F[d(u,u')] \leq a_1F[d(u,u')] + a_2F[d(u,u)] + a_3F[d(u',u')] + a_4F[d(u,u')] + a_5F[d(u',u)]$

$= (a_1 + a_4 + a_5) F[d(u,u')]$

$\leq (a_1 + a_2 + a_3 + a_4 + a_5) F[d(u,u')]$

But $S \geq 1$, so we have:

$F[d(u,u')] \leq (a_1 + a_2 + a_3 + Sa_4 + Sa_5) F[d(u,u')]$

$F[d(u,u')] \leq (a_1 + a_2 + a_3 + S(a_4 + a_5)) F[d(u,u')]$

Since $a_1 + a_2 + a_3 + S(a_4 + a_5) < 1$, so by lemma (2.16) we have $F[d(u,u')] = 0$ and so $d(u,u') = 0$ (i.e.), $u = u'$. Therefore, *f* and *g* have a unique point of coincidence.

Moreover, if the pair (*f* ,*g*) is weakly compatible then by proposition (2.19), u is unique common fixed point of *f* and *g*.

Now we have the following corollaries:

Corollary (3.2):

Let (X,d) be a Cone-b-metric space with the coefficient $S \geq 1$. Suppose the mappings $f, g: X \longrightarrow X$ satisfy for all $x, y \in X$:

$d(fx,fy) \leq a_1d(gx,gy) + a_2d(fx,gx) + a_3d(fy,gy) + a_4d(fx,gy) + a_5d(fy,gx)$ …(3.1.4)

where the constant $a_i \in [0,1)$ and $a_1 + a_2 + a_3 + S(a_4 + a_5) < 1$, $i = 1,2,3,4,5$. If $f(X) \subset g(X)$ and $f(X)$ is complete, then *f* and *g* have a unique point of coincidence. Furthermore if the pair (*f*,*g*) is weakly compatible pair then *f* , *g* have a unique common fixed point.

Proof:

By taking $F(t) = t$ for all $t \in P$, we obtain the required result.

Corollary (3.3):

Let (X,d) be a complete Cone-b-metric space with the coefficient $S \geq 1$. Suppose the mappings $f, g: X \longrightarrow X$ satisfy for all $x, y \in X$:

$d(fx,fy) \leq a_1 d(x,y) + a_2 d(fx,x) + a_3 d(fy,y) + a_4 d(fx,y) + a_5 d(fy,x)$ ...(3.1.5)

where the constant $a_i \in [0,1)$ and $a_1 + a_2 + a_3 + S(a_4 + a_5) < 1$, $i = 1,2,3,4,5$. Then f has a unique fixed point in X.

Proof:

By taking $F(t) = t$ for all $t \in P$ and taking $g(x) = x$ for all $x \in X$, we obtain the required result.

The following corollary is theorem (2.3) of [10].

Corollary (3.4):

Let (X,d) be a complete Cone-b-metric space with the coefficient $S \geq 1$. Suppose $f: X \longrightarrow X$ be a mapping satisfy for all $x, y \in X$:

$d(fx, fy) \leq a_2 d(fx,x) + a_3 d(fy,y) + a_4 d(fx,y) + a_5 d(fy,x)$ ...(3.1.6)

where the constant $a_i \in [0,1)$ and $a_2 + a_3 + S(a_4 + a_5) < \min\{1, \frac{2}{5}\}$, $i = 2,3,4,5$. Then f has a unique fixed point in X.

Proof:

By taking $F(t) = t$ for all $t \in P$ and $g(x) = x$ for all $x \in X$, also by taking $a_1 = 0$ in theorem (3.1) we obtain the required result.

The following corollary is theorem (2.1) in [10] .

Corollary (3.5):

Let (X,d) be a complete Cone-b-metric space with the coefficient $S \geq 1$. Suppose $f: X \longrightarrow X$ be a mapping satisfy for all $x, y \in X$:

$d(fx, fy) \leq a_1 d(x,y)$ ...(3.1.7)

where the constant $a_i \in [0,1)$. Then f has a unique fixed point in X.

Proof:

By taking $F(t) = t$ for all $t \in P$ and $g(x) = x$ for all $x \in X$, also by taking $a_2=a_3=a_4=a_5=0$ in theorem (3.1) we obtain the required result.

Corollary (3.6):

Let (X,d) be a Cone-b-metric space with the coefficient $S \geq 1$. Suppose the mappings $f, g:: X \longrightarrow X$ satisfy for all $x, y \in X$:

$F[d(fx, fy)] \leq a_1 F[d(gx,gy)] + \lambda\{F[d(fx,gx)] + F[d(fy,gy)]\} + \beta\{F[d(fx,gy) + F[d(fy,gx)]\}$ …(3.1.8)

where the constants $a_1, \lambda, \beta \in [0,1)$ with $a_1 + 2\lambda + 2S\beta < 1$ and F be altering function. If $f(X) \subset g(X)$ and $f(X)$ is complete, then f and g have a unique point of coincidence. Furthermore if the pair (f,g) is weakly compatible pair then f, g have a unique common fixed point.

Proof:

By taking $\lambda = a_2 = a_3$ and $\beta = a_4 = a_5$ in theorem (3.1) we obtain the required result.

Corollary (3.7):

Let (X,d) be a complete Cone-b-metric space with the coefficient $S \geq 1$. Suppose the mappings $f, g:: X \longrightarrow X$ satisfy for all $x, y \in X$:

$F[d(fx, fy)] \leq a_4 F[d(fx,gy)] + a_5 F[d(fy,gy)]$ … (3.1.11) where the constants $a_i \in [0,1)$ and $S(a_4 + a_5) < 1$, i = 4, 5 and F be Cone-b-altering function. If $f(X) \subset g(X)$ and $f(X)$ is complete, then f and g have a unique point of coincidence. Furthermore if the pair (f,g) is weakly compatible pair then f, g have a unique common fixed point.

Proof:

By taking $a_1 = a_2 = a_3 = 0$ in theorem (3.1) we obtain the required result.

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بعض مبرهنات نقاط التطابق و النقاط الصامدة المشتركة لتطبيقين ذاتيين يحققان شرطاً منكمشاً معمماً في فضاء قرصي متري من النوع (b)

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المستخلص

في هذا البحث، تم اثبات بعض مبرهنات نقاط التطابق و النقاط الصامدة المشتركة لزوج من تطبيقات ذاتية غير مستمرة متبادلة تبادلاً ضعيفاً يحققان شرطاً منكمشاً معمماً في فضاء قرصي متري من النوع (b) مع افتراض ان القرص المستخدمة في هذا الفضاء غير طبيعية. نتائجنا هي تعميم لبعض النتائج الحالية.

الكلمات المفتاحية : نقاط التطابق والنقاط الصامدة المشتركة ، زوج من التطبيقات المتبادلة الضعيفة، تطبيق ذاتي منكمش معمم ، فضاء قرص المتري من الصنف b، قرص طبيعي، قرص غير طبيعي .