



On The Asymptotic Action of the Difference Equations Class

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Abstract

This paper investigates the global behavior and periodicity of solutions for a broad class of higher-order nonlinear difference equations arising in discrete dynamical systems. The study aims to establish rigorous sufficient conditions ensuring the existence and uniqueness of positive solutions under appropriate initial conditions. Using advanced techniques from nonlinear analysis, the qualitative properties of the solutions are examined, including boundedness, persistence, and convergence. Particular emphasis is placed on the asymptotic stability of equilibrium points, where explicit criteria are derived to determine the conditions under which solutions converge to steady states. The periodic behavior of solutions is also analyzed, with special attention given to the existence of prime period-two solutions under suitable parameter constraints. These results reveal the complex and rich dynamical structure that such nonlinear systems can exhibit. Furthermore, the long-term behavior of solutions is studied in detail, highlighting their convergence patterns and stability characteristics over time. To support the theoretical analysis, several numerical examples are presented, illustrating the applicability and accuracy of the obtained results. These examples clearly demonstrate both the asymptotic stability and periodic dynamics predicted by the theoretical findings, thereby confirming the validity and effectiveness of the proposed approach.

Keywords: Periodic solutions; Asymptotically difference equation; Difference Equations, Locally asymptotically stable, Prime Period two solutions.

1. Introduction

A variety of phenomena can be modeled using difference equations, which are essential mathematical tools, especially in systems with discrete-time dynamics. Numerous scientific disciplines, such as biological sciences, physics, and economics, have shown a great deal of interest in investigating of the asymptotic structure of outcomes to these equations. This interest stems from the reality that many artificial and natural systems are intrinsically discrete and change gradually over time, rendering difference equations a crucial tool for simulating such systems. Recent developments in computational methods have made it possible to solve difference equations more accurately and efficiently, which has led to new understandings of the behavior of the systems these equations describe. However, from a theoretical standpoint, there is still much to learn about stability, regularity, and equilibrium points in the complex and rich field of asymptotic behavior of solutions.

Due to its importance in comprehending the long-term actions of discrete dynamical systems, the asymptotic structure of difference formulas has been extensively studied. The motion of more complex nonlinear difference formulas was first studied¹, setting the stage for later research on their stability as well as long-term behavior. Building on these principles, research² concentrated on finding periodic results in non-linear difference equations, providing useful techniques for examining periodicity in systems with motion. The dynamics of second-order rational difference

formulas were further examined³, revealing unresolved issues and hypotheses concerning their periodic behavior and long-term stability. The understanding of second-order mathematical equations and their worldwide motion has greatly benefited from this study. The difference equation $x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}$ was studied⁴ in the same year helping to investigate asymptotic behavior and stability in nonlinear equations involving difference. In the mid-2000s⁵ looked into the periodicity of max-type difference equation solutions and determined the prerequisites for their existence. The concept of regularity in nonlinear difference formulas was further developed⁶, which investigated the form of outcomes and regularity for third-order rational equations with differences. A major development in the study of rational difference equations occurred when⁷ established a new class of rational difference calculations and investigated how they behave in terms of stability and periodicity. Further investigation of the characteristics and actions of the results for these problems in their subsequent work⁸ significantly advanced our knowledge of asymptotic responses to difference equations. In a similar vein,⁹ made by a substantial contribution to the study of nonlinear rational difference formulas by offering a thorough analysis of these equations, dealing with the asymptotic constancy and regularity of their solutions¹⁰ carried on this investigation, extending the theory to incorporate new characteristics of these equations' solutions and conducting a more thorough analysis of their long-term behavior. Moving on to more recent studies,¹¹ examined the changing dynamics for the difference equation $x_{n+1} = p_n + \frac{x_{n-k}}{x_n}$, using a period-two coefficient, deepening our knowledge of the periodicity and stability of difference equations via periodical coefficients¹² investigated the asymptotic behavior of rational difference equations, concentrating on the equation $y_{n+1} = A + \frac{y_{n-k}}{y_n}$, helping to analyze the stability of such systems. Further developments were made¹³, who analyzed the recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^p}$, focusing on non-oscillatory solutions and their asymptotic behavior in nonlinear difference equations. Expanded on this work, providing a note on the positive no oscillatory solutions of the difference equation $x_{n+1} = \alpha + \frac{x_{n-k}^p}{x_n^p}$, and discussing their long-term stability¹⁴. The convergence of solutions for the difference equation $x_{n+1} = 1 + \frac{x_{n-1}}{x_n}$ was investigated, adding valuable insights into the asymptotic stability of this specific form of difference equations^{15,16} explored the recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}$, addressing the stability and asymptotic behavior of solutions in recursive sequences¹⁷ also contributed to the understanding of positive no oscillatory solutions for difference equations of the form $x_{n+1} = \alpha + \left(\frac{x_{n-k}}{x_{n-m}}\right)^p$, providing important results on the stability of these solutions. For more details see¹⁸⁻²⁴.

By providing more sophisticated methods and findings in the area of difference equations, more recent studies like¹⁰'s investigation into the global actions of nonlinear difference formulas of higher order and²⁵, which discussed the novel approach to obtain periodic outcomes of rational difference formulas, continue to build on these fundamental contributions. These investigations, which concentrate on the stability, periodicity, and asymmetric behavior of difference equations, have made a substantial theoretical contribution.

The purpose of this paper is to better understand the long-term behavior of the group of difference equations by examining their asynchronous nature. We investigate the stability and regularity of solutions using a combination of computational methods and well-known theorems, offering fresh findings that advance our knowledge of difference formulas in dynamic systems. This work aims to improve the conceptual structure and practical usage of difference equations by investigating equilibrium locations, periodic outcomes, and asynchronous stability of these equations.

The goal of this paper is to look into the behavior of a broad class asymptotically different equation solutions.

$$x_{\xi+1} = \sum_{i=0}^g a_i x_{\xi-\beta}^i x_{\xi-\mu}^{r-i}, \xi = 0, 1, \dots \quad (1)$$

whenever $r, a_i, i = 0, 1, \dots, m$ are nonnegative genuine numbers, β, μ and m is nonnegative integer numbers and the initial data $x_{-s}, x_{-s+1}, \dots, x_0$ are positive real numbers where $s = \max\{\beta, \mu\}$.

In this work, we consider the following higher-order nonlinear difference equation:

$$x_{\xi+1} = \sum_{i=0}^g a_i x_{\xi-\beta}^i x_{\xi-\mu}^{r-i}, \xi = 0, 1, \dots$$

This equation represents a general class of nonlinear recursive relations in which the future state of the system depends on multiple delayed terms with nonlinear interactions. Such formulations naturally arise in the modeling of discrete-time processes where memory effects and time delays play a significant role, including applications in population dynamics, economics, and engineering systems. The presence of multiple delays and nonlinear exponents increases the complexity of the system, leading to a wide range of possible dynamical behaviors, including stability, oscillations, and periodicity. Therefore, analyzing this type of equation provides valuable insights into the qualitative behavior of nonlinear discrete systems and contributes to the broader understanding of their long-term dynamics.

1.1. Definition (equilibrium point). Examine the following difference equation:

$$x_{\xi+1} = \theta(x_{\xi-\mu}, x_{\xi-\beta}), \xi = 0, 1, \dots \quad (2)$$

where θ is a continuous function, which be μ, β a positive-integer, if the element $\hat{x}$ is a fixed-point of θ , then it considered an equilibrium point of the **Equation 2**. That is; $\hat{x} = \theta(\hat{x}, \hat{x})$.

1.2. Definition (Stability) Consider the equilibrium-point of **Equation 2** to be $\bar{x} \in (0, \infty)$. Thus, we conclude the types following:

- i- (Local stability) A locally stable equilibrium points $\hat{x}$ of **Equation 2** is regarded as that if for every $\varepsilon > 0$ there exists $\sigma > 0$ in that, if $x_{-c} \in (0, \infty)$ for $c = 0, 1, \dots, r$; $r = \max\{\beta, \mu\}$ with $\sum_{i=0}^r |x_{-i} - \hat{x}| < \sigma$. Then $|x_{\xi} - \hat{x}| < \varepsilon$ for all $\xi \geq -r$.
- ii- (Local asymptotic stability) A locally asymptotically stable equilibrium point $\hat{x}$ of **Equation 1** is called as that if it locally stable and there is $\gamma > 0$ implies that, if $x_{-i} \in (0, \infty)$ for $i = 0, 1, \dots, r$ via $\sum_{i=0}^r |x_{-i} - \hat{x}| < \sigma$.
- iii- (Global stability) A global attractor equilibrium-point $\hat{x}$ of formula (2) is called that if for each $x_{-i} \in (0, \infty)$ for $i = 0, 1, \dots, r$, we deduce $\lim_{\xi \rightarrow +\infty} x_{\xi} = \hat{x}$.
- iv- (Unstability) An equilibrium-points $\hat{x}$ of (2) is regarded as un-stable if it not stables locally.

1.3. Definition (Periodicity) $\{x_{\xi}\}_{\xi=-r}^{\infty}$ is a regarded as periodic via period t if $x_{\xi+t} = x_{\xi}$, $\forall \xi \geq -r$, and the sequence $\{x_{\xi}\}_{\xi=-r}^{\infty}$ is called a periodic via prime period t in the case t is positive integer and smallest possessing this sufficient condition.

1.4. Definition The linear difference equation given by **Equation 2** provides the linearized equation concerning the equilibrium point $\hat{x}$:

$$z_{\xi+1} = \sum_{i=0}^r k_i z_{\xi-i},$$

were

$$k_i = \frac{\partial F(\hat{x}, \hat{x}, \dots, \hat{x})}{\partial x_{\xi-i}}, i = 0, 1, \dots, r$$

1.5. Theorem Assume that $k_j \in R, j = 0, 1, \dots, r$. Then

$$\sum_{j=0}^r |k_j| < 1,$$

meets the necessary requirements for **Equation 2** to be asymptotically stable³.

1.6. Theorem Make I an interval of genuine numerals and state that $Q: I \times I \rightarrow T$

is a continuous function that meets the requirements listed below.

a- $\theta(b, c)$ is non-decreasing in u for each $c \in I$, and is nonincreasing in c for each $b \in I$.

b- In case $(g, G) \in I \times I$ is a solution of the system

$$G = Q(G, g), g = Q(g, G), \text{ then } g = G.$$

Thus, there is only one equilibrium, x , for (1.2), and all of its solutions converge to $\hat{x}$.

2. The Stability of the Equation

Here, we examine both the local and global stability of **Equation 1**'s equilibrium point.

Equation 1's positive equilibrium point is provided by

$$\hat{x} = A\hat{x}^r, \text{ where } A = \sum_{i=0}^g a_i$$

If $r \neq 1$, then we find

$$\hat{x} = 0 \text{ or } \hat{x} = A^{1/1-r}$$

Moreover, **Equation 1** only has a zero equilibrium point if $r = 1$ and $A \neq 1$.¹

2.1. Theorem. Presume $0 < r < 1$. For that reason, Asymptotically, **1**'s positive equilibrium point is locally stable.

Proof. Assume that the function $f(b, c): (0, \infty)^2 \rightarrow (0, \infty)$ is a continuous function defined by

$$f(b, c) = \sum_{i=0}^g a_i b^i c^{r-i} \quad (3)$$

Then, we get

$$\frac{\partial f}{\partial b}(b, c) = \sum_{i=0}^g i a_i b^{i-1} c^{r-i}$$

And

$$\frac{\partial f}{\partial c}(b, c) = \sum_{i=0}^g (r - i) a_i b^i c^{r-i-1}$$

Hence, we obtain

$$\frac{\partial f}{\partial b}(\hat{x}, \hat{x}) = \hat{x}^{r-1} \sum_{i=0}^g i a_i,$$

and

$$\frac{\partial f}{\partial c}(\hat{x}, \hat{x}) = \hat{x}^{r-1} \sum_{i=0}^g (r - i) a_i$$

Now, regarding $\hat{x}$, the linearized Equation of (1) is

$$Z_{\xi+1} - \lambda Z_{\xi-\beta} - \gamma Z_{\xi-\mu} = 0,$$

where,

$$\lambda = \frac{\partial f}{\partial b}(\hat{x}, \hat{x}) \text{ and } \gamma = \frac{\partial f}{\partial c}(\hat{x}, \hat{x})$$

2.2. Theorem dictates that (2) is asymptotically stable in case $|\lambda| + |\gamma| < 1$, and so

$$\left| \hat{x}^{r-1} \sum_{i=0}^g i a_i \right| + \left| \hat{x}^{r-1} \sum_{i=0}^g (r - i) a_i \right| < 1$$

Thus, we find

$$\sum_{i=0}^g r a_i < \hat{x}^{1-r}$$

and hence,

$$r \sum_{i=0}^g a_i < A$$

Then, we have that $0 < r < 1$, therefore, the result is achieved.

2.3. Example. Examine the equation of difference as follows:

$$x_{\xi+1} = x_{\xi-1}^{\frac{1}{2}} + 2x_{\xi}x_{\xi-1}^{-\frac{1}{2}} + x_{\xi}^2x_{\xi-1}^{-\frac{3}{2}}, \xi = 0, 1, \dots$$

Note that, $\beta = 0, \mu = 1, r = 1 = 2$ and $g = 2$. Utilizing Theorem 2.1, the positive equilibrium-points of (1) $\hat{x} = (a_0 + a_1 + a_2)^2$ is locally asymptotically stable (**Figure 1**).

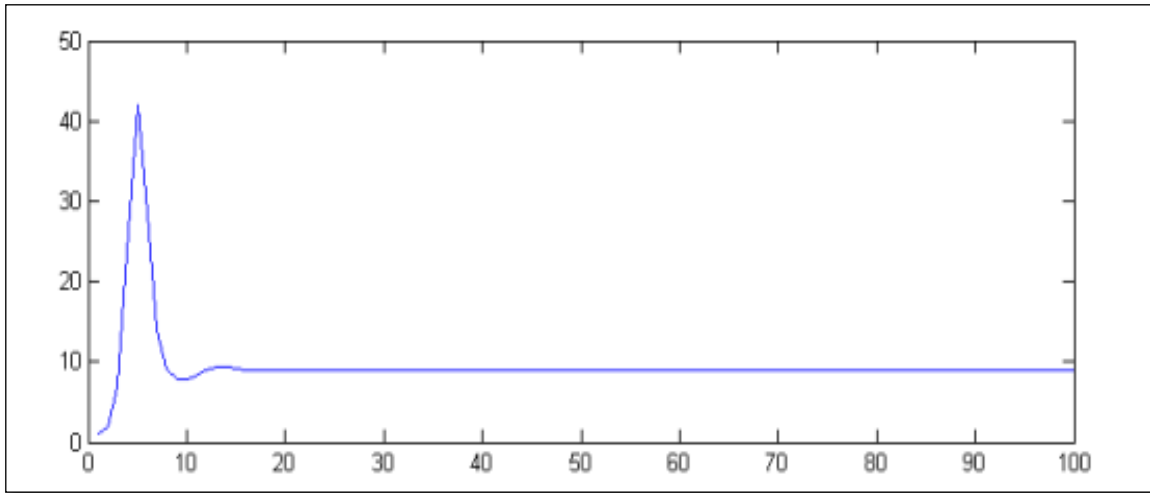


Figure 1. The stable solution that matches Equation 3 for differences

The graph in **Figure 1** represents the evolution of x_{ξ} over time as the system reaches the stable equilibrium. Initially, the values of x_{ξ} may fluctuate, but as shown in the figure, the system stabilizes around the equilibrium point $\hat{x} = (a_0 + a_1 + a_2)^2$, confirming the local asymptotic stability of the solution.

2.4. Theorem Assume that $a_0 = 0$ and $r < 1$. Then, the equilibrium-points $\hat{x}$ is one of equality (1) global attractor.

Proof. Assume that $r < 1$, then we see that the function f defined as (3) satisfies $\frac{\partial f}{\partial b}(b, c) > 0$, and $\frac{\partial f}{\partial c}(b, c) < 0$.

Next, we claim (g, G) as one of the system solutions

$$g = f(g, G), G = f(G, g)$$

Thus, we obtain

$$g = \sum_{j=1}^g a_j g^j G^{r-j}$$

$$G = \sum_{j=1}^g a_j G^j g^{r-j}$$

then,

$$g - G = \sum_{j=1}^g a_j \left(\frac{g^j G^r}{G^j} - \frac{G^j g^r}{g^j} \right),$$

and hence,

$$(g - G) - \sum_{j=1}^g a_j (Gg)^{r-j} (g^{2j-r} - G^{2j-r}) = 0 \quad (4)$$

Hence, $m = M$. By assumption $\hat{x}$ is a global attractor of equality (1.1), Theorem 1.2 implies this fact. Thus, the result is achieved.

3. The Existence of Periodic Solutions

The presence of periodic solutions of **Equation 1** is examined in this section. The sufficient and necessary conditions for periodic solutions of prime period two to exist for this equation are given by the following theorems.

3.1. Theorem If μ, β odd or μ, β even, then there is not a prime period two solution for **Equation 1**.

Proof. Assume that there exists a prime period two solution to equality (1) for l, k , and even.

..., $\varphi, \omega, \varphi, \omega, \dots$

Then $x_{\xi-\mu} = x_{\xi-\beta} = \omega$, from 1, we find

$$\varphi = \omega^r \sum_{i=0}^g a_i$$

$$\omega = \varphi^r \sum_{i=0}^g a_i$$

Thus, we get

$$\varphi = \varphi^{r^2} \sum_{i=0}^g a_i^{r+1}$$

$$\omega = \omega^{r^2} \sum_{i=0}^g a_i^{r+1}$$

Hence, we find

$$\varphi = \omega \sum_{i=0}^g a_i^{\frac{1}{1-r}}$$

which contradicts itself. In a similar manner, the other case may be established, which is left out for convenience.

3.2. Theorem. Suppose that μ even and β is odd. There are a prime period two solutions to formula (1).

..., $\varphi, \omega, \varphi, \omega, \dots$

If

$$\sum_{i=0}^g a_i (T^{1+r-i} - T^i) = 0 \quad (5)$$

where $T = \frac{\varphi}{\omega}$.

Proof: Without loss of generality, it is presupposed that $\beta > \mu$. Here, make equality 1 have a prime period two solution

..., $\varphi, \omega, \varphi, \omega, \dots$

Since β is odd and μ is even, we have $x_{\xi-\beta} = \varphi$ and $x_{\xi-\mu} = \omega$. From **Equation 1**, we get

$$\varphi = \sum_{i=0}^g a_i \varphi^i \omega^{r-i}$$

$$\omega = \sum_{i=0}^g a_i \omega^i \varphi^{r-i}$$

This implies

$$\varphi = \omega^r \sum_{i=0}^g a_i T^i$$

$$\omega = \varphi^r \sum_{i=0}^g a_i T^{-i}$$

where $T = \varphi / \omega$. Since $\varphi = T\omega$, we find

$$\frac{\varphi}{\omega} = \frac{\omega^r \sum_{i=0}^g a_i T^i}{\varphi^r \sum_{i=0}^g a_i T^{-i}}$$

and so,

$$T = T^{-r} \frac{\sum_{i=0}^g a_i T^i}{\sum_{i=0}^g a_i T^{-i}}$$

Then,

$$\sum_{i=0}^g a_i (T^{1+r-i} - T^i) = 0.$$

3.3. Theorem. Suppose that μ is odd and β is even, there is a prime period of two solutions for formula (1.1)

$$\dots, \varphi, \omega, \varphi, \omega, \dots$$

If

$$\sum_{i=0}^g a_i (T^{1+r-i} - T^i) = 0, \quad (6)$$

where $T = \frac{\varphi}{\omega}$.

Proof: We can presume that $\beta > \mu$ without losing generality. Let us now consider a prime period two solution to **Equation 1**.

$$\dots, \varphi, \omega, \varphi, \omega, \dots$$

Since β even and μ odd, we have $x_{\xi-\beta} = \omega$ and $x_{\xi-\mu} = \varphi$. From **Equation 1**, we get

$$\varphi = \sum_{i=0}^g a_i \omega^v \varphi^{r-i}$$

$$\omega = \sum_{i=0}^g a_i \varphi^i \omega^{r-i}$$

This implies

$$\varphi = \varphi^r \sum_{i=0}^g a_i T^{-i}$$

$$\omega = \omega^r \sum_{i=0}^g a_i T^i$$

where $T = \frac{\varphi}{\omega}$. Since $\varphi = T\omega$, we find

$$\frac{\varphi}{\omega} = \frac{\varphi^r \sum_{i=0}^g a_i T^{-i}}{\omega^r \sum_{i=0}^g a_i T^i},$$

and so,

$$T = T^r \frac{\sum_{i=0}^g a_i T^{-i}}{\sum_{i=0}^g a_i T^i},$$

Then,

$$\sum_{i=0}^g a_i (T^{1-r+i} - T^{-i}) = 0.$$

4. Illustrative Numerical Examples and Results

In this section, we give some numerical examples to apply Theorem 3.2 first.

4.1. Example: Consider the equation of difference.

$$x_{\xi+1} = a_0 x_{\xi-1}^2 + a_1 x_{\xi} x_{\xi-1} + a_2 x_{\xi}^2 \quad (7)$$

We note that $m = r = 2$, $\beta = 0$ and $\mu = 1$. **Equation 7** has a prime period two solution i , by applying Theorem 3.2, if

$$a_0(T^{-1} - 1) + a_1(1 - T^{-1}) + a_2(T - T^{-2}) = 0$$

and so,

$$a_0(T - 1)T + a_1(T - 1)T = a_2(T^3 - 1)$$

Since $T \neq 1$, we obtain

$$a_0 - a_1 = \frac{T^2 + T + 1}{T} a_2$$

Now, we see that

$$H(T) = \frac{T^2 + T + 1}{T} > \min_{T \in \mathbb{R}^+} H(T) = 3 \text{ for } T \in \mathbb{R}^+ - \{1\}$$

Then, we obtain that $(a_0 - a_1) > 3a_2$. For numerical example, $a_0 = 8$, $a_1 = 1$, $a_2 = 2$, $x_{-1} = \frac{1}{9}$ and $x_0 = \frac{1}{18}$, show it in **Figure 2**.

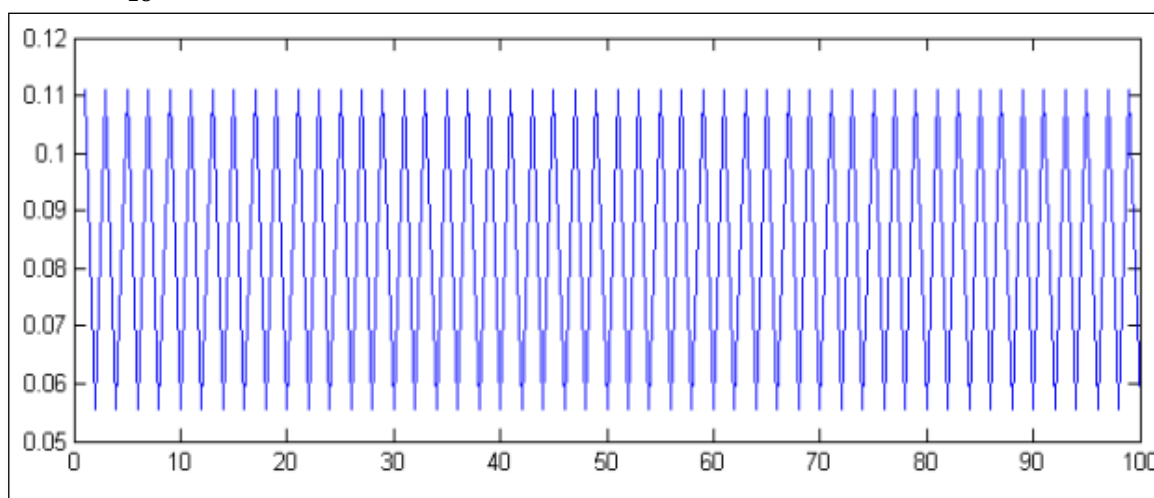


Figure 2. Prime period two solution for equality 7.

The figure shows the periodic solution of a prime period of two for the difference equation:

$x_{\xi+1} = a_0 x_{\xi-1}^2 + a_1 x_{\xi} x_{\xi-1} + a_2 x_{\xi}^2$ with given values $a_0 = 8$, $a_1 = 1$, $a_2 = 2$ and initial conditions $x_{(-1)} = \frac{1}{9}$ and $x_0 = \frac{1}{18}$.

4.2. Example Regard the difference equation

$$x_{\xi+1} = (cx_{\xi} + x_{\xi-1})^g \quad (8)$$

$$= \sum_{i=0}^g \binom{g}{i} c^i x_{\xi}^i x_{\xi-1}^{g-i}$$

where c is a positive real number and m is positive integer, note that, $r = g \geq 1$. By using Theorem 2.1, in (1), the positive equilibrium point is unstable in (8) has a prime period two solution according to Theorem 3.3,

if $\sum_{i=0}^g \binom{g}{i} c^i (T^{1-g} T^i - T^{-i}) = 0$,

and it leads to,

$$T^{1-g} \sum_{i=0}^g \binom{g}{i} c^i T^i = \sum_{i=0}^g \binom{g}{i} c^i \left(\frac{1}{T}\right)^i$$

Thus, we deduce

$$T^{1-g} (cT + 1)^g = \left(\frac{c}{T} + 1\right)^g,$$

Hence,

$$C = \frac{T - T^{1/g}}{T^{(\frac{1}{g})+1} - 1}.$$

Now, we see that

$$H(T) = \frac{T - T^{1/g}}{T^{(\frac{1}{g})+1} - 1} < \min_{T \in \mathcal{R}^+} H(T) = \frac{g-1}{g+1}$$

for $T \in \mathcal{R}^+ - \{1\}$

Therefore, we reach that

$$g > c(g + 1) + 1.$$

For numerical example, $g = 3, c = 0.46825, x_{-1} = 0.80449$ and $x_0 = 0.26816$, shown in **Figure 3**.

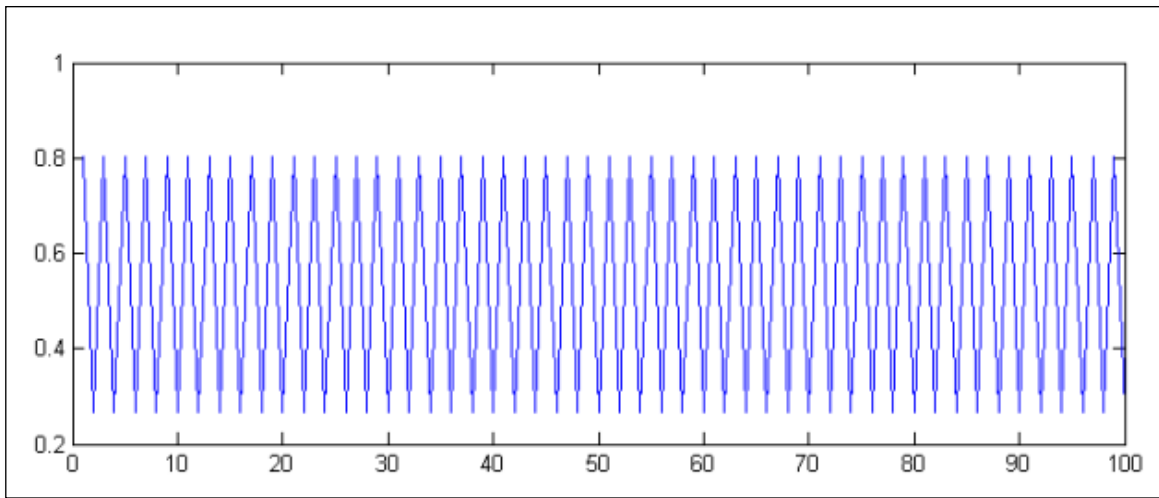


Figure 3: Solution of Equation 8 for prime period two.

The graph shows the oscillatory behavior of x_ξ over time. It demonstrates that the values of x_ξ repeat periodically every two steps, confirming the prime period two solution. The graph exhibits regular fluctuations between two distinct values, which is characteristic of periodic solutions in difference equations. The oscillations persist over time, indicating the stability and periodicity of the system.

4.3. Results

In this section, we present the numerical results derived from applying Theorem 3.2 and other theoretical findings to the difference equations under study. Specifically, we focus on the prime period two solutions for several difference equations and demonstrate their stability and periodicity through numerical simulations.

Example 1: Prime Period Two Solution for Equation 7

We begin by analyzing the difference equation:

$$x_{(\xi+1)} = a_0 x_{(\xi-1)}^2 + a_1 x_\xi x_{(\xi-1)} + a_2 x_\xi^2$$

with parameters $a_0 = 8, a_1 = 1$, and $a_2 = 2$, and initial conditions $x_{(-1)} = \frac{1}{9}$ and $x_0 = \frac{1}{18}$.

Utilizing Theorem 3.2, we confirm that the equation exhibits a prime period two solution. This periodic behavior is observed in Figure 2, where the values of x_ξ oscillate periodically with a

period of two, verifying the theoretical prediction.

Example 2: Prime Period Two Solution for **Equation 8**

Next, we examine the difference equation:

$$x_{(\xi+1)} = a_0 x_{(\xi-1)}^2 + a_1 x_{\xi} x_{(\xi-1)} + a_2 x_{\xi}^2$$

with the following parameters: $g = 3$, $c = 0.46825$, $x(-1) = 0.80449$, and $x_0 = 0.26816$. The graph shown in Figure 3 demonstrates the periodic solution for the prime period two case. The solution oscillates between two distinct values at regular intervals, confirming that the system behaves as predicted by Theorem 3.3, where the values of x_{ξ} repeat every two steps.

4.4. Stability and asymptotic behavior

The periodic solutions observed in both examples are confirmed to be locally asymptotically stable, as the system's state settles into a periodic cycle after initial fluctuations. The stability of the solutions is a critical aspect of this study, as it demonstrates that, despite initial disturbances, the system converges to a periodic solution with consistent oscillations between two values. This stability is a direct consequence of the mathematical properties described in the theorems and is verified through numerical simulations.

5. Conclusion

The manuscript aims to determine the periodicity of results for certain types of difference equations. **Equation 1** provides existence and uniqueness of linear difference equations via some specific conditions. For more, we provided the prerequisites to the existence of periodic solutions; the paper analyzes and studies the asymptotic movement of the solutions, concentrating on the stability and periodicity. Finally, we make in this study some numerical examples with asymptotic stability in the modules of difference equations.

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Conflict of Interest

Non-conflict.

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