Statistical Approximation Operators

Sahib K. Jassim
Zainab I. AbdulNaby
Department of Mathematics / College of Science/Al–Mustansiriya University

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Abstract

In this paper we obtain some statistical approximation results for a general class of max-product operators including the paused linear positive operators.
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Introduction

Let (x_n) be sequences of numbers. Then, (x_n) is called statistically convergent to a number L iff or every$\varepsilon > 0$,$\lim_j \frac{\#\{n\le j: |x_n - L| \ge \epsilon\}}{j} = 0$ .Where $\#$ B denoted the cardinality of the subset of B [1]. We denote this statistical limit by $st - \lim_j x_n = L$. Now, let $A = (a_{j,n})$ be a finite summability matrix. Then, the A-transform of x, denoted by $Ax = (Ax)_j = \sum_{n=1}^{\infty} a_{j,n} x_n$, and provided the series converges for each j. We say that A is a regular if $\lim_j((Ax)_j) = L$, where ever $\lim_j x_j = L$ [2].Assume now that A is nonnegative regular summability matrix. Then, a sequence (x_n) is said to be A-statistically convergent to L if, for every > 0 ,

$\lim_j \sum_{n:|x_n - L| \ge \varepsilon} a_{j,n} = 0$
…(1.1)

It is denoted by $st - \lim_j x_n = L$, [2].

Properties of A Statistical

We recall some basic properties of A-statistical convergence as follows:

1) Let$K$ be a subset of N, the set of all natural numbers. The A- density of K, denoted by$\delta_A(K)$, is defined by$\delta_A(K) = \lim_j \sum_{n\epsilon K} a_{j,n} \chi_K(n)$, provided that the limit exists, where $\chi_K$the characteristic function of isK .By (1.1), we easily see that in [3], $st - \lim_j x_n = L$ if and only if$\delta_A(K) = (\{n: |x_n - L| \ge \varepsilon\}) = 0$, for every$\varepsilon > 0$.

2) Every convergent sequence is A- statistically convergent to the same value for any non-negative regular matrixA , but its converse is not true. For example in [3], for the sequence $x = (x_n)$, is defined as $x_k = \begin{cases} 1, & if\ k\ is\ squar \\ 0, & otherwise. \end{cases}$

It is easy to see that $st - \lim x_k = 0$.

Let $X = [0, \infty)$ and $C(X)$ is the space of all real-valued continuous functions on X and $C_B(X) = \{f \in C(X): f$ is bounede on $X\}$ with the norm on $C_B(X)$ is given by:

$\|f - g\| = \bigvee_{x\in X} |f(x) - g(x)|$ …(2.1)

Let $f: C(X) \to C_B(X)$ be bounded, let also $x_k \in X, k \in \{0, \dots, n\}, n \ge 1$ be fixed sampled data. Then the general discrete form of a max-product approximation is given by [3]:

$$P_n(f, x) = \bigvee_{k=0}^{n} K_n(x, x_k).\ f(x_k)$$

Where $K_n(x, x_k): C(X) \to C_B(X)$, $k = 0, \dots, n$. Are continuous functions onXsuch that,

$\delta_A(\{n \in N: \bigvee_{k=0}^{n} K_n(x, x_k) = 1\}) = 1$ …(2.2)

Let X be normed space and $f: X \to [0, \infty)$be a target. Let $x_k \in X$,$y_k = f(x_k)$ be sampled data. Then the Shepard-type max-product operator associated to f is define by [3]:

$S_n^{\lambda}(f, x) = \bigvee_{k=0}^{n} K_n(x, x_k).\ f(x_k)$ …(2.3)

Where the Shepard Kernel for $x \ne x_k$ is:

$$K_n(x, x_k) = \frac{\frac{1}{\|x - x_k\|^{\lambda}}}{\bigvee_{k=0}^{n} \frac{1}{\|x - x_k\|^{\lambda}}} = 1 \quad \dots(2.4)$$

so the max-product Shepard operator is:

$$S_n^{\lambda}(f, x) = \bigvee_{k=0}^{n} \frac{\frac{1}{\|x - x_k\|^{\lambda}}}{\bigvee_{k=0}^{n} \frac{1}{\|x - x_k\|^{\lambda}}}.\ f(x_k) = \frac{\bigvee_{k=0}^{n} \frac{f(x_k)}{\|x - x_k\|^{\lambda}}}{\bigvee_{k=0}^{n} \frac{1}{\|x - x_k\|^{\lambda}}} \quad \dots(2.5)$$

We put

$Z = \{n \in N: \bigvee_{k=0}^{n} K_n(x, x_k) = 1\}$ …(2.6)

So by (2.2), we may write that

$\delta_A(Z) = 1$ and $\delta_A(N/_Z) = 0$ …(2.7)

Then we define the$T_n(f,x)$ max-product operators from $C(X)$ into $C_B(X)$ as [1]:

$T_n(f,x) = (1+u_n)S_n^{\lambda}(f,x),\ x\epsilon X$ …(2.8)

Where u_n is a convergent sequence. In order to study approximation properties of the operator defined above in (2.8) we need the following definition and lemmas.

Definition 1: [4]

Let $f(x)$ be defined on an interval I and suppose that we can find two positive constants M and α , such that$|f(x_1) - f(x_2)| \le M|x_1 - x_2|^{\alpha}$, for all $x_1, x_2 \in I$ then f is said to satisfy Lipschitz condition of order α and we say that $f \in lip(\alpha)$.

Lemma 2.1: [5]

For any functions $a_k, b_k \in [0,\infty), k \in \{0,\dots,n\}$, we have:

$$\left|\bigvee_{k=0}^{n} a_k - \bigvee_{k=0}^{n} b_k\right| \le \bigvee_{k=0}^{n} |a_k - b_k|$$

Now we prove lemma (2.2), since using in our work.

Lemma 2.2:

Let $x \in X$ and $f \in C(X,[0,\infty))$be fixed. Then the inequality:

$$|f(y) - f(x)| \le \varepsilon + \frac{2M_f}{\delta^2}\varphi_x(y)$$

is valid for sufficiently Largent, where $M_f = \bigvee\{|f(y)|, y \in X\}$ and$\varphi_x(y) = d^2(x,y)$.

Proof:

Let $x, y \in X = [0,\infty)$, and by uniformly continuity of $f$on $[0,\infty)$, for each $\in > 0$ there exists a number $\delta > 0$ such that$|f(x) - f(y)| < \epsilon$, where $\sqrt{x^2 - y^2} < \delta$. Now let $x, y \in [0,\infty)$ and let $z \in [0,\infty)$ such that$0 \le z < \infty$, since f is continuous on the boundary point also for each $\epsilon > 0$ there exists a $\delta > 0$ such that,

$$|f(x) - f(y)| \le |f(x) - f(z)| + |f(z) - f(y)| < \epsilon$$

Where $\sqrt{x^2 - y^2} < \delta$, and$\delta = inf_{y\in X}\ d(x,y)$.

Finally let $x, y \in [0,\infty)$ and let $\sqrt{x^2 - y^2} > \delta$, then easy calculations show that and by lemma (2.1)we have $|f(x) - f(y)| \le M|x - y|$

By using the notation Euclidean n-space R^n in [6],$n \ge 1$is equipped with the distance

$$|x - y| = (\sum_{i=1}^{n}(x_i - y_i)^2)^{\frac{1}{2}}.$$

Then we have that,

$$|f(x) - f(y)| \le M|x - y|$$

$$\le M\sqrt{(x-y)^2}$$

$$\le 2\ M_f \frac{|x^2 - y^2|}{\delta^2}$$

$$\le M_f \frac{2}{\delta^2} d^2(x,y)$$

$$\le \epsilon + \frac{2M_f}{\delta^2} d^2(x,y)$$

$$\le \epsilon + \frac{2M_f}{\delta^2} \varphi_x(y)$$

Theorem 2.1:

Let $T_n, n \in N$ be the linear positive operator in (2.8), which are mappings from C(X) into $C_B(X)$, is continuous in the uniform distance and it is pseudo linear in the sense that

$$T_n(\alpha.f \vee \beta.g) = \alpha.T_n(f,x) \vee \beta.T_n(g,x)$$

Proof:

By (2.8), (2.5), we have

$$|T_n(f,x) - T_n(g,x)| = \left|(1+u_n)S_n^{\lambda}(f,x) - (1+u_n)S_n^{\lambda}(g,x)\right|$$
$$=\left|(1+u_n)\bigvee_{k=0}^{n} K_n(x,x_k).\ f(x_k) - (1+u_n)\bigvee_{k=0}^{n} K_n(x,x_k).\ g(x_k)\right|$$

By lemma (2.1)

$$|T_n(f,x) - T_n(g,x)| \le (1+u_n)\bigvee_{k=0}^{n} K_n(x,x_k)\,|f(x_k) - g(x_k)|$$

By (2.2)

$$|T_n(f,x) - T_n(g,x)| \le (1+u_n)\bigvee_{k=0}^{n} K_n(x,x_k)\,\|f-g\|$$

Since $K_n(x,x_k)$ are continuous and bounded then we have

$$|T_n(f,x) - T_n(g,x)| \le M\|f-g\|$$

Where $M = (1+u_n)\bigvee_{k=0}^{n} K_n(x,x_k)$ the pseud linearity of T_n is obvious.

Theorem 2.2:

Let $T_n, n \in N$, be a positive linear operators from C(X) into $C_B(X)$ be an arbitrary normed space and let $A = (a_{j,n})$ be non-negative regular summability matrix.

If $st_A - \lim_n x_n\{\vee\{|T_n(\varphi_x,x)|: x \epsilon X\}\} = 0$, with $\varphi_x(y) = d^2(x,y)$ …(2.9)

Then, for all $f: C(X) \to C_B(X)$, we have

$$st_A - \lim_n x_n\left\{\bigvee\{|T_n(f,x) - f(x)|: x \epsilon X\}\right\} = 0$$

Proof:

$$|T_n(f,x) - f(x)| = \left|(1+u_n)S_n^{\lambda}(f,x) - f(x)\right|$$
$$= \left|(1+u_n)\bigvee_{k=0}^{n} K_n(x,x_k).f(x_k) - f(x)\right|$$

By $K_n(x,x_k) = \dfrac{\frac{1}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n}\frac{1}{\|x-x_k\|^{\lambda}}} = 1$ in (2.4) we get

$$|T_n(f,x) - f(x)| = \left|(1+u_n)\bigvee_{k=0}^{n} K_n(x,x_k).f(x_k) - \bigvee_{k=0}^{n} K_n(x,x_k).f(x)\right|$$

$$|T_n(f,x) - f(x)| = \left|(1+u_n)\left(\bigvee_{k=0}^{n}\frac{\frac{1}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n}\frac{1}{\|x-x_k\|^{\lambda}}} f(x_k)\right) - \bigvee_{k=0}^{n}\frac{\frac{1}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n}\frac{1}{\|x-x_k\|^{\lambda}}}.f(x)\right|$$

$$= \left|(1+u_n)\frac{\bigvee_{k=0}^{n}\frac{f(x_k)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n}\frac{1}{\|x-x_k\|^{\lambda}}} - \frac{\bigvee_{k=0}^{n}\frac{f(x)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n}\frac{1}{\|x-x_k\|^{\lambda}}}\right|$$

$$= \left|\frac{\bigvee_{k=0}^{n}\frac{f(x_k)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n}\frac{1}{\|x-x_k\|^{\lambda}}} + u_n\left(\bigvee_{k=0}^{n}\frac{\frac{f(x_k)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n}\frac{1}{\|x-x_k\|^{\lambda}}}\right) - \frac{\bigvee_{k=0}^{n}\frac{f(x)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n}\frac{1}{\|x-x_k\|^{\lambda}}}\right|$$

$$= \left|u_n\frac{\bigvee_{k=0}^{n}\frac{f(x_k)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n}\frac{1}{\|x-x_k\|^{\lambda}}} + \frac{\bigvee_{k=0}^{n}\frac{f(x_k)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n}\frac{1}{\|x-x_k\|^{\lambda}}} - \frac{\bigvee_{k=0}^{n}\frac{f(x)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n}\frac{1}{\|x-x_k\|^{\lambda}}}\right|$$

$$\leq \left| u_n \frac{\bigvee_{k=0}^{n} \frac{f(x_k)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n} \frac{1}{\|x-x_k\|^{\lambda}}} \right| + \left| \frac{\bigvee_{k=0}^{n} \frac{f(x_k)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n} \frac{1}{\|x-x_k\|^{\lambda}}} - \frac{\bigvee_{k=0}^{n} \frac{f(x)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n} \frac{1}{\|x-x_k\|^{\lambda}}} \right|$$

$$\leq \left| u_n \frac{\bigvee_{k=0}^{n} \frac{f(x_k)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n} \frac{1}{\|x-x_k\|^{\lambda}}} \right| + \left| \frac{\bigvee_{k=0}^{n} \frac{f(x_k)-f(x)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n} \frac{1}{\|x-x_k\|^{\lambda}}} \right|$$

By lemma (2.3)

$$\leq \left| u_n \frac{\bigvee_{k=0}^{n} \frac{f(x_k)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n} \frac{1}{\|x-x_k\|^{\lambda}}} \right| + \left| \frac{\bigvee_{k=0}^{n} \frac{\epsilon + \frac{2M_f}{\delta^2}\varphi_x(x_k)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n} \frac{1}{\|x-x_k\|^{\lambda}}} \right|$$

$$\leq \left| u_n \frac{\bigvee_{k=0}^{n} \frac{f(x_k)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n} \frac{1}{\|x-x_k\|^{\lambda}}} \right| + \left| (\epsilon + \frac{2M_f}{\delta^2}). \frac{\bigvee_{k=0}^{n} \frac{\varphi_x(x_k)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n} \frac{1}{\|x-x_k\|^{\lambda}}} \right|$$

Since $|f(x_k)| < |\varphi_x(x_k)|$ for all $x \in X, k = 1, \dots, n$ we have

$$< \left| u_n \frac{\bigvee_{k=0}^{n} \frac{\varphi_x(x_k}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n} \frac{1}{\|x-x_k\|^{\lambda}}} \right| + \left| (\epsilon + \frac{2M_f}{\delta^2}). \frac{\bigvee_{k=0}^{n} \frac{\varphi_x(x_k)}{\|x-x_k\|^{\lambda}}}{\bigvee_{k=0}^{n} \frac{1}{\|x-x_k\|^{\lambda}}} \right|$$

By (2.4), (2.5) and lemma (2.2) we have

$$\leq |u_n \bigvee_{k=0}^{n} K_n(x, x_k).\ \varphi_x(x_k)| + \left| ((\epsilon - 1) + 1 + \frac{2M_f}{\delta^2}). \bigvee_{k=0}^{n} K_n(x, x_k). \varphi_x(x_k) \right|$$

$$\leq (|1 + u_n| + (\epsilon - 1) + \frac{2M_f}{\delta^2}) \bigvee_{k=0}^{n} K_n(x, x_k). \varphi_x(x_k)$$

$$\leq (\epsilon - 1 + \frac{2M_f}{\delta^2})(1 + u_n) \bigvee_{k=0}^{n} S_n^{\lambda}(f, x). \varphi_x(x_k)$$

$$\leq (\epsilon - 1) + \frac{2M_f}{\delta^2} T_n(\varphi_x, x)$$

Now, taking maximum over $x \in X$, in the last inequality gives, for all $n \in K$, that

$\bigvee\{|T_n(f, x) - f(x)|: x \in X\} \leq (\epsilon - 1) + \frac{2M_f}{\delta^2} \bigvee\{|T_n(\varphi_x, x): x \in X|\}$ …(2.10)

For a given $r > 0$, choose an $\in -1 > 0$ such that $\in -1 < r$, and then define the sets

$D : \{n \in N : (\bigvee\{|T_n(f, x) - f(x)|: x \in X\}) \geq r\}$,

$$D^{/} : \left\{ n \in N : (\bigvee\{|T_n(\varphi_x, x) - f(x)|: x \in X\}) \geq \frac{(r - \epsilon + 1)\delta^2}{2M_f} \right\}$$

The inequality (2.7), implies $D \cap Z \subseteq D^{/} \cap Z$ which yields for every $j \in N$, that

$\sum_{n \in D \cap Z} a_{jn} \leq \sum_{n \in D^{/} \cap Z} a_{jn} \leq \sum_{n \in D^{/}} a_{jn}$ …(2.11)

Taking limit as $j \to \infty$ on the both-sides of the inequality (2.11), we get

$\lim_j \sum_{n \in D \cap Z} a_{jn} = 0$ …(2.12)

On the other hand, since

$$\sum_{n \in D} a_{jn} = \sum_{n \in D \cap Z} a_{jn} + \sum_{n \in D \cap (N/Z)} a_{jn} \leq \sum_{n \in D \cap Z} a_{jn} + \sum_{n \in (n/Z)} a_{jn}$$

Holds for every $j \in N$, letting $j \to \infty$ in the last inequality and using (2.12) and also the fact that $\delta_A(N/Z) = 0$, we have $\lim_j \sum_{n \in D} a_{jn} = 0$

This means that

$$\text{st} - \lim_n \left\{ \bigvee\{|T_n(f, x) - f(x)|: x \in X\} \right\} = 0$$

Reference

1- Dumen O. (2005) k-positive linear operators, TOBB University of economics and Technology faculty of art and sciences, pp. 569-576.

2- Agratini O. (2006) On approximation of functions by positive linear operators, stud.Cercet.Stiint, Ser.Mat, proceeding sof ICMI 45, Bacau, pp.17-28.

3- Dirik F. (2007) Statistical convergence and rate of convergence of a sequence of positive Linear operators, Faculty of sciences and arts, Sinop University, 57000 Sinop, Turkey, pp. 147-153.

4- Heinonen J. (2004) Lecturec on Lipchitz analysis, lecturec at the 14th Jyvaskyla summer school, NSF grant DMS 0353544 and DMS0244421, pp.1-77.

5- Bede[1] B.; Nobuhara[2] H.; Dankova[3] M. and Nola[4] A. di. (2007) Approximation by pseudo-linear operator Preprint submitted to Elsevier Science, pp.1-26.

6- CLuckers[1] R.; Comte[2] G. and Loesoer[3] F. (2008), Lipchitz continuity properties, Modnet Barcelona conference, pp.1-26.

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الخلاصه

في هذا البحث حصلنا على بعض النتائج في التقريب الاحصائي في فضاء جداء المؤثرات العام متضمنة المؤثرات شبة الخطية الموجبة.
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