On Solution of Regular Singular Initial Value Problems

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Abstract

This paper devoted to the analysis of regular singular initial value problems for ordinary differential equations with a singularity of the first kind , we propose semi - analytic technique using two point osculatory interpolation to construct polynomial solution, and discussion behavior of the solution in the neighborhood of the regular singular points and its numerical approximation, two examples are presented to demonstrate the applicability and efficiency of the methods**.** Finally , we discuss behavior of the solution in the neighborhood of the singularity point which appears to perform satisfactorily for singular problems.

Kay ward : ODE, IVP ,Singular initial value problems.

Introduction

In the study of nonlinear phenomena in physics, engineering and other sciences, many mathematical models lead to singular initial value problems (SIVP) associated with nonlinear second order ordinary differential equations (ODE) [1].

In mathematics, a singularity is in general a point at which a given mathematical object is not defined, or a point of an exceptional set where it fails to be well-behaved in some particular way, such as many problems in varied fields as thermodynamics, electrostatics, physics, and statistics give rise to ordinary differential equations of the form :

$$y'' = f(t, y, y') \quad , \quad a < x < b \qquad (1)$$

On some interval of the real line with some initial conditions.

An IVP associated to the second order differential equation (1) is singular if one of the following situations occurs :

a and/or b are infinite; f is unbounded at some $x_0 \in [0,1]$ or f is unbounded at some particular value of y or y′ [1] .

How to solve a linear ODE of the form :

$$A(x)y'' + B(x)y' + C(x)y = 0 \qquad (2)$$

The first thing we do is, rewrite the ODE as :

$$y'' + P(x)y' + Q(x)y = 0 \qquad (3)$$

where, of course, $P(x) = B(x) / A(x)$, and $Q(x) = C(x) / A(x)$.

there are two types of a point $x_0 \in [0,1]$: Ordinary Point and Singular Point. Also, there are two types of Singular Point : Regular and Irregular Points, A function y(x) is analytic at x_0 if it has a power series expansion at x_0 that converges to y(x) on an open interval containing x_0 .A point x_0 is an ordinary point of the ODE (3), if the functions P(x) and Q(x) are analytic at x_0. Otherwise x_0 is a singular point of the ODE,

i.e. $$P(x) = P_0 + P_1(x-x_0) + P_2(x-x_0)^2 + \ldots\ldots = \sum_{i=0}^{\infty} p_i (x - x_0)^i \qquad (4)$$

$$Q(x) = Q_0 + Q_1(x-x_0) + Q_2(x-x_0)^2 + \ldots\ldots\ldots = \sum_{i=0}^{\infty} q_i (x - x_0)^i \qquad (5)$$

If A , B and C are polynomials then a point x_0 such that $A(x_0) \neq 0$ is an ordinary point. On the other hand if P(x) or Q(x) are not analytic at x_0 then x_0 is said to be a singular. A singular point x_0 of the ODE (3) is a regular singular point of the ODE if the functions x P(x) and x^2 Q(x) are analytic at x_0. Otherwise x_0 is an irregular singular point of the ODE [2] .

Shampine in [3] gave other definition, which illustrated by the following :

If $\lim_{x\to x_0} (x - x_0)P(x)$ finite and $\lim_{x\to x_0} (x - x_0)^2 Q(x)$ finite (6)

that is, if both $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ possess a Taylor series at x_0, then x_0 is called a regular singular point, otherwise x_0 is an irregular singular point .

If A, B and C are polynomials and suppose $A(x_0) = 0$, then x_0 is a regular singular point if :

$$\lim_{x \to x_0} (x - x_0)(B/A) \quad \text{and} \quad \lim_{x \to x_0} (x - x_0)^2 (C/A) \quad \text{are finite} \qquad (7)$$

Now, we state the following theorem without proof which gives us a useful way of testing if a singular point is regular.

Theorem 1 [4]

If the $\lim_{x\to 0} P(x)$ and $\lim_{x\to 0} Q(x)$ exist, are finite, and are not 0 then x = 0 is a regular singular point. If both limits are 0, then x = 0 may be a regular singular point or an ordinary point. If either limit fails to exist or is $\pm\infty$ then x = 0 is an irregular singular point .

There are four kinds of singularities :

- The first kind is the singularity at the first end point of the interval [0,1] ; i.e. , x = 0 .

- The second kind is the singularity at both ends of the interval [0,1]
- The third kind is the case of a singularity in the interior of the interval;
- The forth and final kind is simply treating the case of a regular differential equation on an infinite interval.

In this paper, we focus on the first kind .

Solution of Second Order SIVP

In this section we suggest semi analytic technique to solve second order SIVP as following, we consider the SIVP :

$$x^m y'' + f(x, y, y') = 0 \qquad (8a)$$

$$y(0) = A \ , \ y'(0) = B \qquad (8b)$$

where f are in general nonlinear functions of their arguments .

The simple idea behind the use of two-point polynomials is to replace y(x) in problem (8), or an alternative formulation of it, by osculator interpolation polynomials of order 2n+1, P_{2n+1} which enables any unknown boundary values or derivatives of y(x) to be computed .

The first step therefore is to construct the P_{2n+1} , to do this we need the Taylor coefficients of y (x) at x = 0 :

$$y = a_0 + a_1 x + \sum_{i=2}^{\infty} a_i x^i \qquad (9)$$

where $y(0)= a_0$,$y'(0)= a_1$,$y''(0) / 2! =a_2$,…, $y^{(i)}(0) / i! = a_i$, i= 3, 4,…
then insert the series forms (9) into (8a) and put x = 0 and equate coefficients of powers of x to obtain a_2 . Also we need Taylor coefficient of y(x) about x = 1 :

$$y = b_0 + b_1 (x-1) + \sum_{i=2}^{\infty} b_i (x-1)^i \qquad (10)$$

where $y(1) = b_0$, $y'(1) =b_1$, $y''(1) / 2! =b_2$,…, $y^{(i)}(1) / i! =b_i$, i = 3,4,…
then insert the series form (10) into (8a) and put x = 1 and equate the coefficients of powers of (x-1) to obtain b_2 ,then derive equation (8a) with respect to x to obtain new form of equation say (11) :

$$x^m y''' + m x^m y'' + df(x, y, y')/dx = 0 \qquad (11)$$

then, insert the series form (9) into (11) and put x = 0 and equate the coefficients of powers of x to obtain a_3 ,then insert the series form (10) into (11) and put x = 1 and equate the coefficients of powers of (x-1) to obtain b_3 , now iterate the above process many times to obtain a_4 ,b_4 ,then a_5 ,b_5 and so on, that is ,we can get a_i and b_i for all $i \geq 2$, the resulting equations can be solved using MATLAB to obtain a_i and b_i for all $i \geq 2$, the notation implies that the coefficients depend only on the indicated unknowns a_0 , a_1, b_0 , b_1, and we get a_0 , a_1, by the initial condition .Now, we can construct a $P_{2n+1}(x)$ from these coefficients (a_i's and b_i's) by the following :

$$P_{2n+1} = \sum_{i=0}^{n} \{ a_i Q_i(x) + (-1)^i b_i Q_i(1-x) \} \qquad (12)$$

where $(x^j / j!)(1-x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s = Q_j(x) / j!$

we see that (12) have only two unknowns b_0 and b_1 to find this, we integrate equation (8a) on [0, x] to obtain :

$$x^m y'(x) - m x^{m-1} y(x) + m(m-1)\int_0^x x^{m-2} y(x)\, dx + \int_0^x f(x, y, y')\, dx = 0 \qquad (13a)$$

and again integrate equation (13a) on [0, x] to obtain :

$$x^m y(x) - 2m\int_0^x x^{m-1} y(x)\,dx + m(m-1)\int_0^x (1-x)x^{m-2} y(x)dx + \int_0^x (1-x)f(x,y,y') = 0 \text{ , (13b)}$$

Putting x = 1 in (13), then gives :

$$b_1 - mb_0 + m(m-1)\int_0^1 x^{m-2}\,y(x)\,dx + \int_0^1 f(x, y, y')\,dx = 0 \qquad (14a)$$

and

$$b_0 - 2m\int_0^1 x^{m-1} y(x)dx + m(m-1)\int_0^1 (1-x)x^{m-2}\,y(x)\,dx + \int_0^1 (1-x)f(x, y, y')dx = 0 \quad (14b)$$

Use P_{2n+1} as a replacement of y(x) in (14) and substitute the initial conditions (8b) in (14) then, we have only two unknown coefficients b_1, b_0 and two equations (14) so, we can find b_1, b_0 for any n by solving this system of algebraic equations using MATLAB, so insert b_0 and b_1 into (12) , thus (12) represents the solution of (8) .

Extensive computations have shown that this generally provides a more accurate polynomial representation for a given n .

Examples

In this section, many examples will be given to illustrate the efficiency, accuracy , implementation and utility of the suggested method. The bvp4c solver of MATLAB has been modified accordingly so that it can solve some class of SIVP as effectively as it previously solved nonsingular IVP.

Example 1

Consider the following SIVP :

$$y'' + (2/x)\,y' - 10\,y = 12 - 10\,x^4 \quad , \quad 0 \le x \le 1$$

I.C. $y(0) = 0$, $y'(0) = 0$, Exact solution is $y(x) = 2\,x^2 + x^4$

It is clear that x = 0, is regular singular point and it is singularity of first kind . Now, we solve this example using semi - analytic technique , From equations (12) we have : $P_5(x) = x^4 + 2x^2$

Olga [6] solve this example using Modification A domian Decomposition method and gives the exact solution .

Example 2

Consider the following SIVP :

$$y'' + (2/x)\,y' + y = 0 \quad , \quad 0 \le x \le 1$$

with IC : $y(0) = 1$, $y'(0) = 0$.

It is clear that x = 0 , is regular singular point and it is singularity of first kind and the exact solution is y(x) = sin(x) / x [7]. Now, we solve this example using semi-analytic technique ,From equation (12) we have :

if n = 6 ,we get P_{13} as follows :

$$P_{13} = -0.0000000000051205\,x^{13} + 0.0000000001756046\,x^{12} - 0.000000000246766464\,x^{11} - 0.000000250276639921x^{10} - 0.0000000000145659x^9 + 0.00000275573675x^8 - 0.00000000000068686\,x^7 - 0.00019841269841336\,x^6 + 0.008333333333333\,x^4 - 0.1666666666678793\,x^2 + 1$$

For more details ,table (1) gives the results of different nodes in the domain, for n= 6. Also, figure(1) illustrated the accuracy of suggested method for n=6.

Ramos[7] solved this example by Linearization techniques and the results gave in figure 2 and the absolute error $E(y) = |y_{exact} - y_{app.}|$ gave in figure 3 for h = 0.01 and 0.001 respectively .

Batiha[8] solved this example by Variational Iteration Method (VIM) and the absolute error between the 5-iterate of VIM and the exact solution gave in table 2.

Behavior of the solution in the neighborhood of the singularity x= 0

Our main concern in this section will be the study of the behavior of the solution in the neighborhood of singular point .

Consider the following SIVP :

$$y''(x) + ((N-1)/x)\, y'(x) = f(y) \ , N \geq 1 \ , \quad 0 < x < 1 \qquad (15)$$

$$y(0) = y_0 \ , \quad \lim_{x\to 0+} x\, y'(x) = 0 \qquad (16)$$

where f(y) is continuous function .

As the same manner in [9], let us look for a solution of this problem in the form :

$$y(x) = y_0 - C\, x^k (1 + o(1)) \qquad (17)$$

$$y'(x) = -\, C\, k\, x^{k-1}(1 + o(1))$$

$$y''(x) = -\, C\, k\, (k-1)\, x^{k-2}(1 + o(1)) \ , \quad x \to 0^+$$

where C is a positive constant and k > 1. If we substitute (17) in (15) we obtain :

$$C = (1/k)\, (f(y_0)\ /N)^{k-1} \qquad (18)$$

In order to improve representation (17) we perform the variable substitution :

$$y(x) = y_0 - C\, x^k (1 + g(x)) \qquad (19)$$

we easily obtain the following result which is similar to the results in [9].

Theorem 2 [9]

For each $y_0 > 0$, problem (15), (16) has, in the neighborhood of x = 0, a unique solution that can be represented by :

$$y(x, y_0) = y_0 - C\, x^k\, (1 + g\, x^k + o(x^k)) \ ,$$

where k, C and g are given by (18) and (19), respectively.

We see that these results are in good agreement with the ones obtained by the method in [9], they are also consistent with the results presented in [10]. In order to estimate the convergence order of the suggested method at x = 0, we have carried out several experiments with different values of n and used the formula :

$$c_{y0} = -\log_2 (|y_0^{n3} - y_0^{n2}| / |y_0^{n2} - y_0^{n1}|) \qquad (20)$$

where y_0^{ni} is the approximate value of y_0 obtained with n_i ,n_i = 1,2, 3, 4,…

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Table 1: The result of the method for n = 6 of example 2

b_0	0.841470984807897		
b_1	- 0.301168678939757		
x_i	Exact solution y(x)	P_{13}	Error | y(x) – P_{13} |
0	1	1	0
0.1	0.998334166468282	0.998334166468282	e-0162204460492503 12.
0.2	0.993346653975306	0.993346653975307	e-0164 40892098500626 .4
0.3	0.985067355537799	0.985067355537799	0
0.4	0.973545855771626	0.973545855771626	1.110223024625157e-016
0.5	0.958851077208406	0.958851077208406	e-01611022302462516.1
0.6	0.941070788991726	0.941070788991726	0
0.7	0.920310981768130	0.920310981768130	0
0.8	0.896695113624403	0.896695113624404	e-016 22044604925031 2.
0.9	0.870363232919426	0.870363232919426	. 33066907387547 e-016 3
1	0.841470984807897	0.841470984807897	0
SSE			e-0314.314083075427408

Table 3: The comparison of the suggested method and VIM [8] of example 2

x_i	Exact solution y(x)	Absolute Error of VIM
0	1	0
0.1	0.998334166468282	1.00000E-20
0.2	0.993346653975306	6.70000E-19
0.3	0.985067355537799	8.53100E-17
0.4	0.973545855771626	2.69223E-15
0.5	0.958851077208406	3.91600E-14
0.6	0.941070788991726	3.48972E-13
0.7	0.920310981768130	2.21760E-12
0.8	0.896695113624403	1.10021E-11
0.9	0.870363232919426	4.51811E-11
1	0.841470984807897	1.59829E-10
SSE		2.771272831459776 e-020

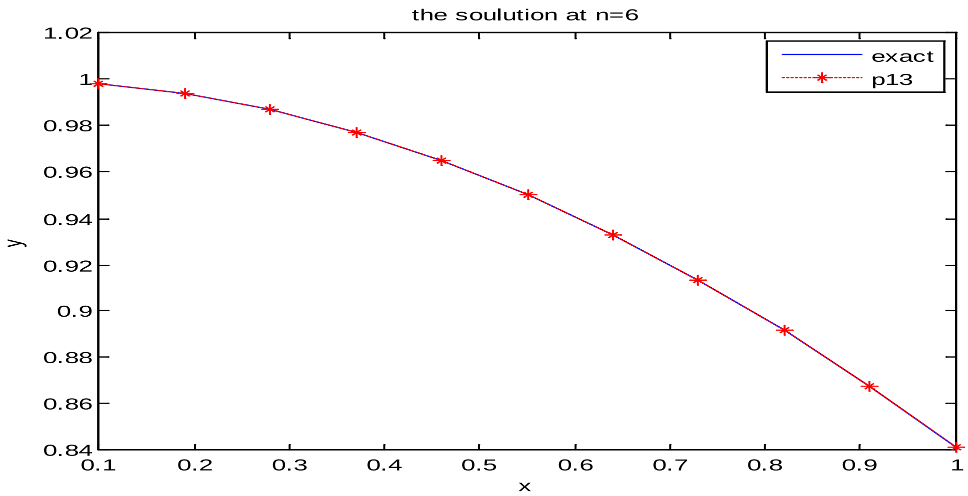

Figure 1: illustrate suggested method for n= 6,i.e.,P_{13} of example 2 .

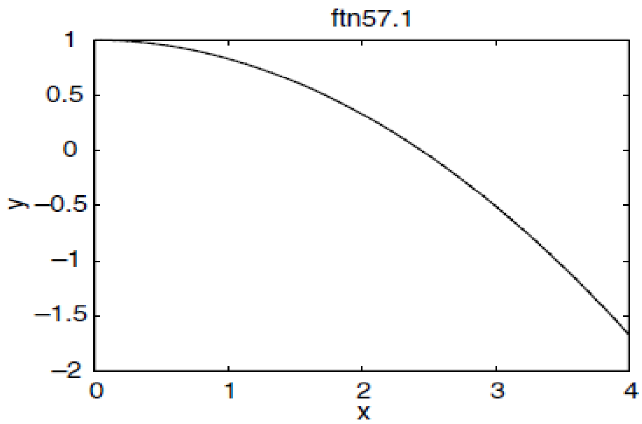

Figure 2: Linearization techniques for example 2 gave in [7]

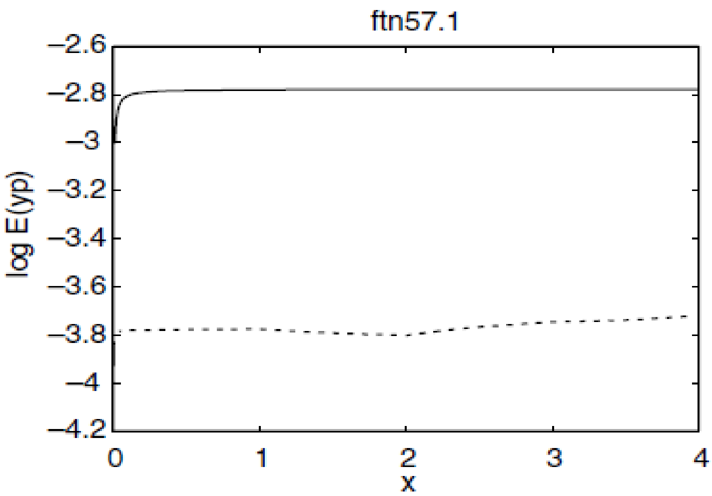

Figure 3: absolute error of method gave in [7] for h=0.01 and 0.001

حول حل مسائل القيم الابتدائية النظامية الشاذة

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الخلاصة

الهدف من هذا البحث عرض دراسة تحليلية لمسائل القيم الابتدائية النظامية الشاذة للمعادلات التفاضلية الاعتيادية وبأنواع مختلفة ،إذ إننا نقترح التقنية شبه التحليلية باستخدام الاندراج التماسي ذي النقطتين للحصول على الحل بوصفها متعددة حدود ومناقشة عدد من الأمثلة لتوضيح الدقة ، والكفاية ، وسهولة أداء الطريقة المقترحة .و أخيرا ناقشنا سلوك الحل في جوار النقاط الشاذة ، و إيجاد الحل التقريبي لها . و اقترحنا صيغة جديدة مطورة لتخمين الخطأ تساعد في تقليل الحسابات العملية وإظهار النتائج بشكل مرضي فيما يخص المسائل الشاذة .

الكلمات المفتاحية : المعادلات التفاضلية الاعتيادية، مسائل القيم الابتدائية ، مسائل القيم الابتدائية الشاذة