On Semi-p-Proper Mappings

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Abstract

The aim of this paper is to introduce a new type of proper mappings called semi-p-proper mapping by using semi-p-open sets, which is weaker than the proper mapping. Some properties and characterizations of this type of mappings are given.

Keywords : Proper mapping, **s**emi-p-open sets.

Introduction

One of the important concepts in topology and in mathematics in general is the concept of mapping, in this paper we give a new class of mappings.

The class of proper mappings was first introduced by Valnsteln in 1947 and studied by Lary in 1950 and Bourbaki in 1951, who defined the proper mapping as (if $f : X \longrightarrow Y$ is continuous and $f \times I_Z$ is closed for each space Z then f is called a proper mapping).
For more details see [1], [2], [3], [4], [5] and [6]. In this paper we introduced and studied semi-p-proper mappings.

2- Semi-p-Open Sets

Throughout this paper X, Y and Z are topological spaces or space only with no separation axioms assumed unless otherwise stated. The interior and closure of a subset A of a topological space will be denoted by intA and clA respectively.

In this section we shall recall some needed definitions, propositions, and properties of semi-p-open sets and pre-open sets which we shall use to define semi-p-proper mapping, some of these propositions or definition are given for the first time by the author.

Definition 2.1:

A subset A of a space X is called

(1) pre-open if $A \subseteq int(clA)$, [7].

(2) semi-p-open if $U \subseteq A \subseteq$ pre-cl U for some pre-open set U in X and pre-cl U is the smallest pre-closed set which contains U, [8].

The complement of pre-open (respectively semi-p-open) set is called a pre-closed (respectively semi-p-closed) set.

Remarks 2.2: [9]

(1) X and ϕ are pre-open (semi-p-open)sets.

(2) The union of any family of pre-open (semi-p-open) set is a pre-open (semi-p-open) set.

(3) The intersection of two pre-open (semi-p-open) sets need not to be pre-open (respectively semi-p-open).

Examples 2.3: [9], [4] and [10]

(1) If X is any infinite set and τ_{co} is the cofinite topology on X, then the family of pre-open sets PO(X) and the family of semi-p-open sets S-PO(X) are equal and PO(X) = S-PO(X) = $\{U \subseteq X : U$ is an infinite set or $U = \phi\}$.

(2) In the Discrete space (X,D) we have
PO(X) = SPO)X) = D = $\mathbb{P}$(X) = the family of all subsets of X.

(3) In the indiscrete space (X,I) where I = $\{X, \phi\}$, PO(X) = SPO(X) = $\mathbb{P}$(X).

(4) In the space $(\mathbb{R},\tau_u)$ where $\mathbb{R}$ is the set of real numbers and τ_u is the usual topology on $\mathbb{R}$, PO(X) = SPO(X) = $\tau_u \cup \{Q, Irr\}$ where Q is the rationals and Irr is the irrationals.

Proposition 2.4: [11]

Let A be any subset of the space X, then

(1) $(intA)^c = clA^c$.

(2) $(clA)^c = intA^c$.

Proposition 2.5: [4]

Let A be any subset of a space X, then A is semi-p-open set iff $A \subseteq$ pre-cl(pre-intA) where pre-clA is the smallest pre-closed set which contains A, and pre-intA is the largest pre-open set contained in A.

Corollary 2.6:

Let A be a subset of a space X. A is semi-p-closed set iff pre-int(pre-clA) $\subseteq$ A.

Proof: (⇒) Let A be a semi-p-closed subset of X, then pre-clA = A, which implies pre-int(pre-clA) $\subseteq$ A since pre-intA $\subseteq$ A.

(⇐) Suppose pre-int(pre-clA) $\subseteq$ A. To prove A is semi-p-closed set.
Since pre-int(pre-clA) $\subseteq$ A, so we get
$A^c \subseteq [\text{pre-int(pre-cl}A)]^c$
$= \text{pre-cl(pre-cl}A)^c$
$= \text{pre-cl(pre-int}A^c)$
Hence A^c is a pre-open set by proposition 2.5 which means A is pre-closed. ■

Corollary 2.7:

Let A be a subset of a space X. A is semi-p-open set iff A $\subseteq$ cl(pre-intA).

Proof: Since pre-clA $\subseteq$ clA and by proposition 2.5. ■

Proposition 2.8: [9]

Let A and B be subsets of a space X, then pre-int(A $\cap$ B) = pre-intA $\cap$ pre-intB.

Proposition 2.9:

Let A be an open set in a space X and B is a semi-p-open set, then A $\cap$ B is a semi-p-open set.

Proof: we shall use corollary 2.7 to prove this proposition; that is we must show that
A $\cap$ B $\subseteq$ cl(pre-int(A $\cap$ B)). Let x $\notin$ cl(pre-int(A $\cap$ B)) which implies
$\exists$ U open in X s.t. x $\in$ U and U $\cap$ pre-int(A $\cap$ B) = ϕ.
Then U $\cap$ (pre-intA $\cap$ pre-intB) = ϕ (by proposition 2.8)
So we get (U $\cap$ pre-intA) $\cap$ pre-intB = ϕ
Hence (U $\cap$ A) $\cap$ pre-intB = ϕ since A is open
Which means U $\cap$ A is open in X
Now if x $\in$ A then x $\in$ U $\cap$ A and (U $\cap$ A) $\cap$ pre-intB = ϕ which means
x $\notin$ cl(pre-intB) but B $\subseteq$ cl(pre-intB), since B is semi-p-open
So x $\notin$ B $\Rightarrow$ x $\notin$ A $\cap$ B. ■

Proposition 2.10:

If A is closed in X and B is semi-p-closed then A $\cap$ B is semi-p-closed set in X.

Proof: $(A \cap B)^c = A^c \cup B^c$ where A^c is open and B^c is semi-p-open so $A^c \cup B^c$ is semi-p-open. ■

Proposition 2.11:

Let A be a pre-open subset of the space X, and let B be a pre-open subset of the space Y then A×B is pre-open in the product space X×Y.

Proof: A $\subseteq$ int(clA) and B $\subseteq$ int(clB), A×B $\subseteq$ int(clA) × int(clB) = int(cl(A×B)). So A×B is pre-open in X×Y. ■

Proposition 2.12:

Let A be a subset of a space X, and B a subset of a space Y then

1) pre-cl(A×B) = pre-clA × pre-clB.

2) Pre-int(A×B) = pre-intA × pre-intB.

Proof: Obvious. ■

Corollary 2.13:

The product of any two semi-p-open sets in X and Y respectively is semi-p-open in X×Y.

Proof: Let A be a semi-p-open subset of X, and let B be a semi-p-open subset of Y, then $A \subseteq$ pre-cl(pre-intA) and $B \subseteq$ pre-cl(pre-intB) by proposition 2.5 which implies
$A\times B \subseteq$ pre-cl(pre-intA) × pre-cl(pre-intB)
= pre-cl(pre-intA×B) by proposition 2.12. ■

Corollary 2.14:

If A is a nonempty subset of X, and B a non empty subset of Y then:

(1) If A is pre-closed subset of X and B is pre-closed subset of Y then A×B is pre-closed subset of X×Y.

(2) If A is a semi-p-closed subset of X and B is a semi-p-closed subset of Y then A×B is semi-p-closed subset of X×Y.

Proof: The proofs of parts (1) and (2) are similar, so we prove part (2) because our work about semi-p-closed sets.

(2) A is semi-p-closed subset of X means pre-int(pre-clA) $\subseteq$ A. By corollary 2.14
On the other hand B is semi-p-closed subset of Y implies pre-int(pre-clB) $\subseteq$ B. So we have pre-int(pre-clA) × pre-int(pre-clB) $\subseteq$ A×B.
Hence pre-int(pre-cl A×B) $\subseteq$ A×B. ■

It known, that a mapping $f: X \longrightarrow Y$ is continuous if the inverse image of each closed set in Y is closed in X.
On the other hand f is an open (closed) mapping if the image of each, open (closed) subset of X is open (closed) in Y, [12].

Now, we shall recall two definitions of a mapping using semi-p-closed sets, these are known by, semi-p-closed and semi-irresolute mappings.

Definition 2.15: [9]

Let $f: X \longrightarrow Y$ be a mapping, f is called:

(1) semi-p-closed if $f(A)$ is semi-p-closed in Y whenever A is closed in X.

(2) semi-p-irresolute if $f^{-1}(B)$ is semi-p-closed in X whenever B is semi-p-closed in Y.

Remarks 2.16: [9]

(1) The composition of two semi-p-closed mappings need not be semi-p-closed in general.

(2) The composition of a semi-p-closed and continuous mappings is semi-p-closed.

(3) Every homeomorphism is a semi-p-closed mapping.

In the following definition we give a new type of homeomorphisms named semi-p-homeomorphism which is weaker than the concept homeomorphism.

Definition 2.17:

A mapping $f: X \longrightarrow Y$ is called semi-p-homeomorphism if it is bijective, continuous and semi-p-closed.

Remark 2.18:

It is clear that every homeomorphism is semi-p-homeomorphism, but not converse.

For example:

Let $f: (\mathbb{R},D) \longrightarrow (\mathbb{R},I)$ be a mapping defined by $f(x) = x \ \forall \ x \in \mathbb{R}$. It is clear that f is not closed, which means that f is not a homeomorphism.

On the other hand f is semi-p-homeomorphism, since it is bijective, continuous and semi-p-closed mapping, since S-PO(ℝ,I) =ℙ(ℝ) by examples 2.3 (3).

Proposition 2.19:

Let X, X′, X″ be three topological spaces, and let $f: X \longrightarrow X'$, $g: X' \longrightarrow X''$ be two mappings, then:

(1) If $g\circ f$ is semi-p-closed and if f is continuous and surjective, then g is semi-p-closed.

(2) If $g\circ f$ is semi-p-closed and if g is semi-p-irresolute and injective, then f is semi-p-closed.

Proof: We have

(1) $X \xrightarrow{f} X' \xrightarrow{g} X''$, with $g\circ f: X \longrightarrow X''$

Let B be a closed subset of X′ to prove g(B) is semi-p-closed in X″. Now B closed in X′ and f is continuous, then f^{-1}(B) is closed in X, but $g\circ f$ is semi-p-closed, so $(g\circ f)(f^{-1}(B))$ is semi-p-closed in X″, but $(g\circ f)(f^{-1}(B)) = g(B)$ since f is surjective, hence g is semi-p-closed.

(2) Let A be a closed subset of X to prove f(A) is semi-p-closed in X′. Since A is closed in X and $g\circ f$ is semi-p-closed, we get $(g\circ f)$(A) is semi-p-closed in X″, but g is semi-p-irresolute, which implies $g^{-1}((g\circ f)(A))$ is semi-p-closed subset of X′, but $g^{-1}((g\circ f)(A)) = f(A)$ since g is injective, which means that f is semi-p-closed mapping. ■

3- Semi-p-Proper Mapping

In this section we shall define a new type of proper mappings named semi-p-proper mapping by using the concept of semi-p-open set which depends on the, pre-open set.

Recall that a mapping f from a topological space X into a topological space Y is called proper if f is continuous and $f \times I_Z : X\times Z \longrightarrow Y\times Z$ is closed for every topological space Z, [3].

Definition 3.1:

Let f be a mapping from a topological space X into a topological space Y, then f is called a semi-p-proper mapping if f is continuous and $f\times I_Z: X\times Z \longrightarrow Y\times Z$ is semi-p-closed for any topological space Z.

Remarks 3.2:

(1) Every proper mapping is semi-p-proper since every closed mapping is semi-p-closed, but the converse is not true in general. For example:

Let ℝ be the set of real numbers with the usual topology then i: Q ⟶ ℝ is semi-p-proper mapping which is not proper because it is not closed, because Q is semi-p-closed subset of ℝ which is not closed.

(2) Let X be any topological space, and F be a closed subset of X, then the inclusion mapping i: F ⟶ X is semi-p-proper since it is a proper mapping, i is proper because it is a closed mapping since F is closed in X and, so every closed subset of F is closed in X.

(3) The identity $I_x: X \longrightarrow X$ is proper and so it is semi-p-proper, I_X is proper since it is 1-1 and a closed mapping..

In the following theorem, we give a characterization for semi-p-proper mappings.

Theorem 3.3:

Let $f: X \longrightarrow Y$ be a continuous injection mapping, then the following statements are equivalant:

(1) f is semi-p-proper.

(2) f is semi-p-closed.

(3) f is semi-p-homeomorphism from X onto a semi-p-closed subset of Y.

Proof:

(1) ⇒ (2) Let f be semi-p-proper. To prove f is semi-p-closed mapping. Take Z = {p} then by hypothesis the mapping $f\times I_Z: X\times\{p\}\longrightarrow Y\times\{p\}$ is semi-p-closed, but X×{p} and Y×{p} are homeomorphic to X and Y respectively, then f is semi-p-closed mapping.

(2) ⇒ (3) f is continuous, injection and semi-p-closed by hypothesis, so it is semi-p-homeomorphism from X onto f(X) which is semi-p-closed subset of Y.

(3) ⇒ (1) Let f be semi-p-homeomorphism mapping from X into a semi-p-closed subset of Y, i.e. $f: X\longrightarrow f(X)$ is semi-p-homeomorphism mapping. Now, let Z be any topological space, so $I_Z: Z\longrightarrow Z$ the identity mapping on Z is homeomorphism, which implies $f\times I_Z: X\times Z\longrightarrow Y\times Z$ is semi-p-homeomorphism from X×Z onto the closed subspace $f(X)\times Z$ of Y×Z and therefore is a semi-p-closed mapping. ■

Proposition 3.4:

The composition of two semi-p-proper mapping is semi-p-proper.

Proof: Let X, X′ and X″ be topological spaces, and let $f: X\longrightarrow X'$ and $g: X'\longrightarrow X''$ be two semi-p-proper mappings. To prove $g\circ f$ is semi-p-proper.

$g\circ f$ is a continuous mapping, since f and g are continuous. Now let Z be any topological space, then $f\times I_Z$ is semi-p-closed mapping since f is semi-p-proper by hypothesis, on the other hand $g\times I_Z$ is continuous since g and I_Z are continuous by [3,p.44], so $(g\times I_Z)\circ(f\times I_Z)$ is semi-p-closed mapping by remarks 2.16 (2) which implies $(g\circ f)\times I_Z$ is semi-p-closed. ■

Proposition 3.5:

Let X, X′ and X″ be three topological spaces, and let $f: X\longrightarrow X'$ and $g: X'\longrightarrow X''$ be mappings, then:

(1) If f is continuous surjective mapping, g is continuous and $g\circ f$ is semi-p-proper, then g is semi-p-proper.

(2) If f is continuous, g is semi-p-irresolute injective mapping and $g\circ f$ is semi-p-proper then f is semi-p-proper.

Proof:

(1) Let Z be any topological space, then $f\times I_Z$ is continuous and onto since f and I_Z are so, on the other hand $(g\circ f)\times I_Z$ is semi-p-closed since $g\circ f$ is semi-p-proper which implies that $g\times I_Z$ is semi-p-closed by proposition 2.19(1).

(2) Le Z be any topological space, then the mapping $g\times I_Z: X'\times Z\longrightarrow X''\times Z$ is semi-p-irresolute injective mapping since g and I_Z are so. On the other hand $(g\circ f)\times I_Z: X\times Z\longrightarrow X''\times Z$ is semi-p-closed since $g\circ f$ is semi-p-proper, which implies that $f\times I_Z: X\times Z\longrightarrow X'\times Z$ is semi-p-closed by proposition 2.19(2). ■

Proposition 3.6:

Let X_1, Y_1, X_2 and Y_2 be topological spaces, and let $f_1: X_1\longrightarrow Y_1$, $f_2: X_2\longrightarrow Y_2$ be mappings, then $f_1\times f_2$ is semi-p-proper if f_1 and f_2 are semi-p-proper mappings.

Proof: Suppose f_1 and f_2 are semi-p-proper mappings to show that $f_1\times f_2$ is semi-p-proper. Let Z be any topological space, then:

$f_1\times f_2\times I_Z = (f_1\times I_{Y_2}\times I_Z)\circ(I_{X_1}\times f_2\times I_Z)$ as shown in the following diagram

$$X_1\times X_2\times Z \xrightarrow{I_{X_1}\times f_2\times I_Z} X_1\times Y_2\times Z \xrightarrow{f_1\times I_{Y_2}\times I_Z} Y_1\times Y_2\times Z$$

$$X_1\times X_2\times Z \xrightarrow{f_1\times f_2\times I_Z} Y_1\times Y_2\times Z$$

but $f_1\times I_{Y_2}\times I_Z$ is continuous since f_1, I_{Y_2} and I_Z are continuous. On the other hand $I_{x_1}\times f_2\times I_Z$ is semi-p-closed since I_{x_1} is semi-p-closed and $f_2\times I_Z$ is semi-p-closed since f_2 is semi-p-proper mapping, hence by remarks 2.16(2) $f_1\times f_2\times I_Z$ is semi-p-closed which means that $f_1\times f_2$ is semi-p-proper. ■

4- Future Work

(1) We can use the concept of semi-p-compactness to characterize semi-p-proper mappings.
(2) We can use the concept of semi-p-proper mapping to define a new kind of spaces called semi-p-proper G-spaces.

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حول الدوال شبه السديدة من النمط – p

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الخلاصة

الهدف من هذا البحث هو اضافة نمط جديد من الدوال السديدة و تسمى الدالة شبه السديدة من نوع – p ، باستخدام المجموعات شبه المفتوحة من نمط-p وهي أضعف من الدوال السديدة. تم برهنة ودراسة بعض خواص هذه الدوال ومكافئاتها.

الكلمات المفتاحية : الدوال السديدة، المجموعات شبه المفتوحة من النمط – p.