Approximation of Functions in $L_{p,\alpha}(I)$ (0 < p < 1)

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Abstract.

In this paper we show that the function $f \in L_{p,\alpha}(I)$,0<p<1 where I=[-1,1] can be approximated by an algebraic polynomial with an error not exceeding $\omega_k^{\varphi}(f, t, \frac{1}{n})_{p,\alpha}$ where $\omega_k^{\varphi}(f, t, \frac{1}{n})_{p,\alpha}$ is the Ditizian–Totik modules of smoothness of unbounded function in $L_{p,\alpha}(I)$

Key Words: Monotone approximation, polynomial, degree of approximation

Introduction

In this paper we study the approximation of unbounded function f in $L_{p,\alpha}(I)$, 0<p<1, where I =[-1,1] by algebraic polynomial, such approximation of unbounded functions has previously been studied in [1] and [2], our main departure from these pervious works is that we shall prove direct estimates for the error of polynomial approximation in terms of the Ditizian Totick modules of smoothness .We shall find these estimates in two cases, the monotone case by monotone algebraic polynomial and unconstrained case, the estimates of modules of smoothness only in first and second order.

Definition and notation

Definition (2.1): [1]

Let f be any function such that $|f(x)| \le Me^{-\alpha x}, \alpha > 0, M \in R^+, x \in [a,b]$ we denote by $L_{p,\alpha}$,the space of all functions f such that ,

$$\|f\|_{p,\alpha} = \left[\int_b^a |f(x)e^{-\alpha x}|^p dx\right]^{1/p} < \infty \quad ,1 \le p < \infty \qquad \text{…(2.1)}$$

Definition (2.2): [1]

If J is an interval then k-th order module of smoothness of f is defined by

$$\omega_k(f,t,J)_{p,\alpha} := \sup_{0\le h\le t}\left(\int_J |\Delta_h^k(f,x,J)e^{-\alpha x}|^p dx\right)^{1/p}, \alpha > 0 \qquad \text{…(2.2)}$$

where Δ_h^k is the symmetric difference, such that

$$\Delta_h^k(f,x,J) := \begin{cases} \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} f\left(x - \frac{k}{2}h + ih\right) & \text{if } x \pm \frac{k}{2}h \in J \\ 0 & \text{otherwise} \end{cases}$$

Definition (2.3): [1]

If $f \in L_{p,\alpha}(I)$ then the Ditizian – Totik modules of smoothness is defined by,

$$\omega_k^{\varphi}(f,t,J)_{p,\alpha} := \sup_{0<h\le t}\left(\int_{-1}^{1} |\varphi(x)\Delta_h^k(f,x,J)e^{-\alpha x}|^p dx\right)^{1/p}, \alpha > 0 , \qquad \text{…(2.3)}$$

where

$$\Delta_{h\varphi(x)}^k(f,x,I) := \begin{cases} \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} f\left(x - \frac{k}{2}h\varphi(x) + ih\varphi(x)\right) & \text{if } x \pm \frac{k}{2}h\varphi(x) \in I \\ 0 & \text{otherwise} \end{cases}$$

Monotone piecewise linear approximation

Let $a = \xi_0 < \xi_1 < ... < \xi_n = b$ such that adjacent $I_j = [\xi_{j-1}, \xi_j], j = 1,...,n$ have comparable lengths, that is

$$\frac{|I_{j\pm1}|}{|I_j|} \le c_0 \qquad \text{...(3.1)}$$

with c_0 an absolute constant. We denote to Y the class of all piecewise linear functions on [a,b] .Each function $S \in$ Y is completely determined by its left and right hand values

$S(\xi_{j\mp}),\ j=1,\dots,n-1$ and the values $S(\xi_0),\ S(\xi_n)$, if f is a function in $L_{p,\alpha}(J)$,$0<p\le 1$,J is an interval then a polynomial p of degree k is a near best $L_{p,\alpha}$ approximation to f from among all polynomials of degree $\le k$ if

$$\|f-p\|_{p,\alpha}(J)\le M\ E_k(f,J)_{p,\alpha} \qquad ...(3.2)$$

Where M is constant and $E_k(f,J)_{p,\alpha}$ is the error of best approximation to f on J in the space $L_{p,\alpha}$ from among all polynomials of degree $\le k$.

Theorem (3.1): [3]

For any interval [a,b] and $a=\xi_0<\xi_1<\dots<\xi_n=b$ there is a piecewise linear function $\bar{S}\in y$ with the following properties :

i) $\bar{S}$ is non decreasing.

ii) There is a constant $M_0>0$ depending only on p and the constant c_0 such that for j=1,…,n, $\bar{\ell}_j$ satisfies (3.2) for an interval $\bar{I}_j$ with $I_j\subset\tilde{I}_j\subset\bar{I}_j:=I_{j-2}\cup I_{j-1}\cup I_j\cup I_{j+1}\cup I_{j+2}$(such that $I_k:=\phi$ if $k\le 0$ and $k>n$).

iii) $\bar{S}(\xi_{0+})\ge S(\xi_{0+})$ and $\bar{S}(\xi_{0-})\le S(\xi_{0-})$

Theorem (3.2): [3]

Under the hypothesis of theorem (3.1) there is a nondecreasing piecewise linear function $S^*\in y$ satisfying (i) and (ii) of theorem (3.1) with $\bar{I}_j$ replaced by $I_j^*=\bigcup_{j-3}^{j+3}I_v$ and the additional property that S^* is continuous.

Definition (3.1):

For $f\in L_{p,\alpha}(I)$ let us define,

$$w_k(f,t,J)_{p,\alpha}:=\left(t^{-1}\int_0^t\int_J\left|\Delta_s^k(f,x,J)e^{-\alpha x}\right|^p dx\,ds\right)^{1/p} \qquad ...(3.3)$$

Lemma (3.1):

For $f\in L_{p,\alpha}$ we have $w_k(f,t,J)_{p,\alpha}\cong\omega_k(f,t,J)_{p,\alpha}$

Proof

$$\omega_k(f,t,J)_{p,\alpha}:=\sup_{0<s\le t}\left(\int_J\left|\Delta_s^k(f,x,J)e^{-\alpha x}\right|^p dx\right)^{1/p},\ \alpha>0$$

$$=\sup_{0<s\le t}\left(t^{-1}t\int_J\left|\Delta_s^k(f,x,J)e^{-\alpha x}\right|^p dx\right)^{1/p}$$

$$\omega_k(f,t,J)_{p,\alpha}\le c\left(t^{-1}\int_0^t\int_J\left|\Delta_s^k(f,x,J)e^{-\alpha x}\right|^p dx\,ds\right)^{1/p}$$

$$=cw_k(f,t,J)_{p,\alpha}$$

$$\omega_k(f,t,J)_{p,\alpha}\le cw_k(f,t,J)_{p,\alpha} \qquad …(3.4)$$

$$w_k(f,t,J)_{p,\alpha} := \left(t^{-1}\int_0^t \int_J \left|\Delta_s^k(f,x,J)e^{-\alpha x}\right|^p dxds \right)^{1/p}$$

$$= \left(t^{-1}\int_0^t (\int_J \left|\Delta_s^k(f,x,J)e^{-\alpha x}\right|^p dx)^{p/p} ds \right)^{1/p}$$

$$\leq \left(t^{-1}\int_0^t ((c \sup_{0<s\leq t} \int_J \left|\Delta_s^k(f,x,J)e^{-\alpha x}\right|^p dx)^{1/p})^p ds \right)^{1/p}$$

$$= \left(t^{-1}\int_0^t (c\,\omega_k(f,t,J)_{p,\alpha})^p ds \right)^{1/p}$$

$$= \left(t^{-1}(cs\,\omega_k^p(f,t,J)_{p,\alpha})\,|_0^t \right)^{1/p}$$

$$= \left(t^{-1}t\,c\,\omega_k^p(f,t,J)_{p,\alpha} \right)^{1/p}$$

$$= c\,\omega_k(f,t,J)_{p,\alpha}$$

Then $w_k(f,t,J)_{p,\alpha} \leq c\,\omega_k(f,t,J)_{p,\alpha}$

and $c^{-1}w_k(f,t,J)_{p,\alpha} \leq \omega_k(f,t,J)_{p,\alpha}$ $\cdots(3.5)$

From(3.4) and (3.5) we get ,

$$c^{-1}w_k(f,t,J)_{p,\alpha} \leq \omega_k(f,t,J)_{p,\alpha} \leq c w_k(f,t,J)_{p,\alpha}$$

This implies that w_k and ω_k are equivalent

$$= \left(t^{-1}\int_0^t ((\int_J \left|\Delta_s^k(f,x,J)e^{-\alpha x}\right|^p dx)^{1/p})^p ds \right)^{1/p}$$

Theorem (3.3): (Whitney's theorem in $L_{p,\alpha}$)

For any $f \in L_{p,\alpha}(I)$ on [0,1] and for any integer $n\geq 1$, there is a polynomial $p(x)$ of degree at most n-1 such that ,

$$\left|(f(x)-p(x))e^{-\alpha x}\right| < 6\,\omega_n(f,\frac{1}{n+1})_{p,\alpha}$$

Let p be the interpolation polynomial of $x_v = \dfrac{v}{n+1}$

If $h = \dfrac{1}{n+1}$, $x = vh + t$, $0 \leq t < h$,

Let $\varphi(f,x) = \varphi(f,vh+t) = \dfrac{(-1)^{n-v}}{h\binom{n}{v}}\displaystyle\int_0^1 \Delta_y^n f(x-vy)e^{-\alpha(x-vy)}dy$

Thus for $x \in [vh,(v+1)h]$,

$$|\varphi(f,x)| \le \frac{1}{\binom{n}{v}}\omega_n(f,\frac{1}{n+1})_{p,\alpha} \qquad ...(3.6)$$

Let $\ell_{n,v}(x)=\prod_{\substack{j=0\\ j\ne v}}^{n}\frac{x-j}{v-j} \quad ,v=0,1,...,n$

Before we prove Whitney theorem in $L_{p,\alpha}$ we need these results.

Proposition (3.2): [6]

Let $P_{n-1}(f)$ be the interpolation polynomial for f at the points $h,2h,\dots,nh$, *i.e.*

$$p_{n-1}(f,x)=\sum_{j=1}^{n}f(jh)\,\ell_{n-1,j-1}(\frac{x}{h}-1), \text{ then}$$

$$f(x)-p_{n-1}=\Delta_h^n f(0)\,\ell_{n,0}(\frac{x}{h})+\varphi_n(f,x)-\sum_{j=0}^{n}\varphi_n(f,x)\ell_{n,j}(\frac{x}{h})+h^{-1}\int_0^t\sum_{j=0}^{n}\varphi_n(f,jh+v)\ell'_{n,j}(\frac{x-v}{h})dv$$

Lemma (3.2): [6]

$$v_n := \max\left\{\sum_{j=0}^{n}\binom{n}{j}^{-1}|\ell_{n,v}(x)|;0\le x\le 1\right\}=1$$

Lemma (3.3): [6]

$$\mu_{n,v}:=\sum_{j=0}^{n}\binom{n}{j}^{-1}\max\{|\ell_{n,v}(x)|;v\le x\le v+1\}\le\frac{1+\sigma_v+\sigma_{v+1}}{\binom{n}{v}} \quad , v=0,1,...,\frac{n-1}{2},$$

and $\sigma_v=1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{v} \quad , \sigma_0=0$

Proposition (3.4)

For $f\in L_{p,\alpha}$ integrable on [0,1] , weget

$$\left|f(x)-\sum_{j=1}^{n}f(jh)\,\ell_{n-1,j-1}(\frac{x}{h}-1)\right|\le\frac{6+7\min(\sigma_v,\sigma_{n-v})}{\binom{n}{v}}\omega_n(f,\frac{1}{h})_{p,\alpha} \quad ,$$

For $x\in[vh,(v+1)h]$, $h=\frac{1}{n+1}$, $v=0,1,...,n$, $\sigma_v=1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{v} \quad , \sigma_0=0$

Proof

Since the interpolation polynomial is ,

$$p_{n-1}(f,x)=\sum_{j=1}^{n}f(jh)\,\ell_{n-1,j-1}(\frac{x}{h}-1),$$

and knots are symmetric with respect to the middle of [0,1], we prove it for $x\in[0,\frac{1}{2}]$.

For v = 0 from lemmas (3.2) and (3.3),using proposition (3.2)and using (3.6) we get,

$$\left|(f(x)-p(x))e^{-\alpha x}\right| \le \omega_n(f,\frac{1}{h})_{p,\alpha}\left[\max_{0\le t\le 1}\left|\ell_{n,0}(t)\right|+1+\max_{0\le t\le 1}\sum_{j=0}^{n}\frac{1}{\binom{n}{v}}\left|\ell_{n,j}(t)\right|+\sum_{j=0}^{n}\frac{1}{\binom{n}{j}}\int_0^t \ell'_{n,j}(v)dv\right]$$

$$\le \omega_n(f,\frac{1}{h})_{p,\alpha}\left[3+1+\sum_{j=0}^{n}\frac{1}{\binom{n}{v}}\max_{0\le t\le 1}\left|\ell_{n,j}(t)\right|\right]$$

$$\le 6\omega_n(f,\frac{1}{h})_{p,\alpha}$$

For $v=1,2,3,\dots,\frac{n-1}{2}$, we have

$$\left|(f(x)-p(x))e^{-\alpha x}\right| \le \frac{\omega_n(f,\frac{1}{h})_{p,\alpha}}{\binom{n}{v}}\left[2+\binom{n}{v}\max_{v\le u\le v+1}\sum_{j=0}^{n}\frac{1}{\binom{n}{j}}\left|\ell_{n,j}(t)\right|+\binom{n}{v}\sum_{j=0}^{n}\frac{1}{\binom{n}{j}}\int_0^t\left|\ell'_{n,j}(v+1)\right|dv\right]$$

$$\le \frac{\omega_n(f,\frac{1}{h})_{p,\alpha}}{\binom{n}{v}}\left\{2+3\binom{n}{v}\sum_{j=0}^{n}\frac{1}{\binom{n}{j}}\max_{v\le u\le v+1}\left|\ell_{n,j}(u)\right|\right\}$$

$$\le \frac{5+3(\sigma_v+\sigma_{v+1})}{\binom{n}{v}}\omega_n(f,\frac{1}{h})_{p,\alpha}$$

$$\le \frac{6+7\sigma_v}{\binom{n}{v}}\omega_n(f,\frac{1}{h})_{p,\alpha}$$

$$\le \frac{6+7\min(\sigma_v,\sigma_{n-v})}{\binom{n}{v}}\omega_n(f,\frac{1}{h})_{p,\alpha}$$

Proof of theorem(3.3)

By using proposition (3.4) we get immediately the proof of theorem (3.3)

For $v=0$

$$\left|(f(x)-p(x))e^{-\alpha x}\right| < 6\omega_n(f,\frac{1}{h})_{p,\alpha}$$

and for $v = 1,2,\dots,\frac{n-1}{2}$ we have

$$\left|(f(x)-p(x))e^{-\alpha x}\right| \le \frac{6+7\min(\sigma_v,\sigma_{n-v})}{\binom{n}{v}}\omega_n(f,\frac{1}{h})_{p,\alpha}$$

Since for every $v = 0,1,2,\dots,n$ we have

$$\frac{6+7\min(\sigma_v,\sigma_{n-v})}{\binom{n}{v}} \le 6$$

Then

$$\left|(f(x)-p(x))e^{-\alpha x}\right| \le 6\omega_n(f,\frac{1}{h})_{p,\alpha}$$

Auxiliary Results

To construct monotone approximation, we shall need properties for the ξ_j .we begin recalling a construction introduced by [4] and used also in [5]. We approximate the truncated functions $\varphi_j(x) := (x-\xi_j)_+$ by algebraic polynomial .If $x=cos\ t$ we obtain algebraic polynomial $r_j(x) := T_{n-j}(t)$.

Since ,

$$T_j(t) := \chi_j * J_n = \int_{t-t_j}^{t+t_j} J_n(u)\,du \qquad j=0,\dots,n \qquad \dots(4.1)$$

where $\chi_j := \chi_{[-t_j,t_j]}$, $t_j := {}^{j\pi}/_{n}$, $j=0,\dots,n$ and J_n be a Jackson kernel which defined by,

$$J_n(t) = \left(\frac{\sin {}^{nt}/_{2}}{{}^{t}/_{2}}\right)^{2r} \quad , \quad \int_{-\pi}^{\pi} J_n(t)\,d = 1 \qquad \dots(4.2)$$

R_j is polynomial of degree $\le nr$ **is** defined by,[3]

$$R_j(x) = \int_{-1}^{x} r_j(u)\,du \qquad \text{and} \quad \theta_j := \chi_{[\xi_j,1]} \qquad j=1,\dots,n \qquad \dots(4.3)$$

Lemma (4.1): [3]

With $\varphi(x) = \sqrt{1-x^2}$ we have for any $n \ge 10$,

i) $c_1\varphi(x)n^{-1} \le \xi_{j+4} - \xi_{j-3} \le c_2\varphi(x)n^{-1}, x \in [\xi_{j-3}, \xi_{j+4}], j = 4,\dots,n-5$

ii) $\left|\xi_j - \cos t_{n-j}\right| \le c(\xi_{j+1} - \xi_j), j = 0,\dots,n-1$

iii) $c_1(\xi_{j+1} - \xi_j) \le \xi_j - \xi_{j-1} \le c_2(\xi_{j+1} - \xi_j), j = 1,2,\dots,n-1$

iv) $1+x \le c_2\varphi(x)n^{-1}, -1 \le x \le \xi_7, and\ 1-x \le c_2\varphi(x)n^{-1}, \xi_{n-7} \le x \le 1$

v) $c_1\varphi(x)n^{-1} \le 1-\xi_{n-1} \quad , \quad \xi_{n-7} \le x \le 1; \ ; \ c_1\varphi(x)n^{-1} \le \xi_1 + 1, -1 \le x \le \xi_7$

where c, c_1 and c_2 are constants dependent of n and x .

Lemma (4.2): [3]

For $j = 1,\dots,n-1$ let $d_j(x) := 1 + \left|x - \xi_j\right| / (\xi_{j+1} - \xi_j)$ then

$$\left|r_j(x) - \theta_j(x)\right| \le c\left[d_j(x)\right]^{-r+1},$$

$$\left|R_j(x) - \varphi_j(x)\right| \le c(\xi_{j+1} - \xi_j)\left[d_j(x)\right]^{-r+2}$$

with the constant c

Lemma (4.3): [3]

If p is polynomial of degree$\le$ k, then for $x \in$ [-1,1] we have,

$$\left|P(x)\right| \le c\left(1 + \frac{\left|x-\xi_j\right|}{\xi_{j+1} - \xi_j}\right)^k \max_{\xi_j \le u \le \xi_{j+1}} \left|P(u)\right|$$

Lemma (4.4): [3]

If $0<p<\infty$ and $k=0,1,\dots,$ then for any polynomial P of degree$\le k$ we have,

$$\max_{a \le x \le b} \left|P(x)\right|^p \le \frac{c}{b-a} \left\|P\right\|_{L_p[a,b]}^p ,$$

for unbounded function we get,

$$\max_{a\le x\le b}\left|P(x)e^{-\alpha x}\right|^p \le \frac{c}{b-a}\|P\|^p_{L_{p,\alpha}[a,b]},$$

where c depends only on k and p.

The Main Result

we fix n=1,2,... and let ξ_j be the points of section 4 we denote by $I_j=[\xi_{j-1},\xi_j]$ and as in the theorem (3.2) $I_j^*=[\xi_{j-4},\xi_{j+3}]\cap I$, $j=1,\dots,n-1$, where $\xi_j:=\xi_0=-1, j\le 0$ and $\xi_j:=\xi_n=1, j\ge n$, for $f\in L_{p,\alpha}(I)$ we denote by $p_j(f)$ an algebraic polynomial of degree $\le k$-1 which is a near best approximation to f on interval I_j^* then from Whitney theorem (3.3) we get ,

$$\|f-p_j(f)\|^p_{p,\alpha}\le c\,\omega_k(f,|I_j|,I_j)^p_{p,\alpha}$$

Let us define the linear operator L_n and P_n by **,**

$$L_n(f,x)=p_1(f,x)+\sum_{j=1}^{n-1}[p_{j+1}(f,x)-p_j(f,x)]\theta_j(x) \qquad \dots(5.1)$$

$$P_n(f,x)=p_1(f,x)+\sum_{j=1}^{n-1}[p_{j+1}(f,x)-p_j(f,x)]r_j(x) \qquad \dots(5.2)$$

Where θ_j and r_j defined in section 4

Theorem (5.1)

If $f\in L_{p,\alpha}$ and k is a positive integer then,

$\|f-L_n(f)\|_{p,\alpha}\le c\,\omega_k^{\varphi}(f,\frac{1}{n})_{p,\alpha}$, where c depends only on k and p

Proof

We fix $n\ge 10$ .For j=5,…,n-4 we have by lemma(4.1) (i) that $c_1\le|I_j^*|/\varphi(x)\le c_2n^{-1}$ for $x\in I_j^*$. Since $L_n(f)=p_j(f)$ on I_j and from Whitney theorem(3.3)and lemma (3.1) we obtain,

$$\begin{aligned}\|f-L_n(f)\|^p_{p,\alpha}(I_j)&\le\|f-p_j(f)\|^p_{p,\alpha}(I_j)\\&\le c\,\omega_k(f,|I_j|,I_j)^p_{p,\alpha}\\&\le c\,\omega_k(f,|I_j^*|,I_j^*)^p_{p,\alpha}\\&\le c\,w_k(f,|I_j^*|,I_j^*)^p_{p,\alpha}\end{aligned}$$

$$\le c\left|I_j^*\right|^{-1}\int_0^{\left|I_j^*\right|}\int_{I_j^*}\left|\Delta_h^k(f,x,I_j^*)e^{-\alpha x}\right|^p dhdx$$

$$= c\int_{I_j^*}\int_0^{\left|I_j^*\right|/\varphi(x)}\frac{\varphi(x)}{\left|I_j^*\right|}\left|\Delta_{h\varphi(x)}^k(f,x,I_j^*)e^{-\alpha x}\right|^p dhdx$$

$$\le c\int_{I_j^*}\int_0^{c_2/n}\frac{n}{c_2}\left|\Delta_{h\varphi(x)}^k(f,x,I_j^*)e^{-\alpha x}\right|^p dhdx$$

$$= cn\int_0^{c_2/n}\int_{I_j^*}\left|\Delta_{h\varphi(x)}^k(f,x,I_j^*)e^{-\alpha x}\right|^p dxdh$$

This inequality also holds for j=1,2,3,4 and j=n-3,n-2,n-1,n .We now sum the inequalities, we obtain

$$\left\|f-L_n(f)\right\|_{p,\alpha}^p(I_j)\le cn\int_0^{c_2/n}\int_I\left|\Delta_{h\varphi(x)}^k(f,x,I)e^{-\alpha x}\right|^p dxdh$$

$$\le cn\int_0^{c_2/n}\sup_{0\le h\le cn^{-1}}\int_I\left|\Delta_{h\varphi(x)}^k(f,x,I)e^{-\alpha x}\right|^p dxdh$$

$$= cn\int_0^{c_2/n}\omega_k^\varphi(f,\frac{1}{n})^p{}_{p,\alpha}dh$$

$$= c\,\omega_k^\varphi(f,\frac{1}{n})^p{}_{p,\alpha}$$

Theorem (5.2):

Let $f\in L_{p,\alpha}$,$0<p<1$ and k be a positive integer then ,for each $n\ge N$ (with a constant depending only on p and k), there exists an algebraic polynomial P_n of degree $\le n$.

$$\left\|f-P_n(f)\right\|_{p,\alpha}\le c\,\omega_k^\varphi(f,\frac{1}{n})_{p,\alpha}$$

where c depends only k and p ,moreover if f is nondecreasing on I ,then for $k\le 2$ the polynomial P_n can also be taken to be nondecreasing .

Proof of theorem (5.2) (part 1) the non constrained case.

$$\left\|f-P_n(f)\right\|_{p,\alpha}=\left\|f-L_n(f)+L_n(f)-P_n\right\|_{p,\alpha}$$

$$\le\left\|f-L_n(f)\right\|_{p,\alpha}+\left\|L_n(f)-P_n\right\|_{p,\alpha}$$

$$\le c\,\omega_k^\varphi(f,\frac{1}{n})_{p,\alpha}+\left\|L_n(f)-P_n\right\|_{p,\alpha}$$

which is defined in (5.2) , is polynomial of degree $\le(n-1)r+k$, $P_n(f,x)$ By theorem (5.1) and the fact

we need only to prove that :

$$\left\|L_n(f)-P_n(f)\right\|_{p,\alpha}\le c\,\omega_k^\varphi(f,\frac{1}{n})_{p,\alpha}$$

Now ,

$$\left\|L_n(f)-P_n(f)\right\|_{p,\alpha}^{p}=\int_I\left|\left[L_n(f,x)-P_n(f,x)\right]e^{-\alpha x}\right|^p dx$$

Then from (5.1), (5.2) and by lemma (4.2) ,lemma (4.3) and lemma (4.4)

$$\int_I\left|\left[L_n(f,x)-P_n(f,x)\right]e^{-\alpha x}\right|^p dx=\int_I\left|\sum_{i=}^{n-1}(p_{i+1}(f,x)-p_i(f,x)(\theta_i(x)-r_i(x))e^{-\alpha x}\right|^p dx$$

$$\le\sum_{i=1}^{n-1}\int_I\left|(p_{i+1}(f,x)-p_i(f,x)e^{-\alpha x})\right|^p\left|\theta_i(x)-r_i(x)\right|^p dx$$

$$\le\sum_{i=1}^{n-1}\int_I\max_{\xi_i<x<\xi_{i+1}}\left|(p_{i+1}(f,x)-p_i(f,x))e^{-\alpha x}\right|^p\left(1+\frac{|x-\xi_i|}{\xi_{i+1}-\xi_i}\right)^{(k-1)p}\left|\theta_i(x)-r_i(x)\right|^p dx$$

$$\le c\sum_{i=1}^{n-1}\max_{\xi_i<x<\xi_{i+1}}\left|(p_{i+1}(f,x)-p_i(f,x))e^{-\alpha x}\right|^p\int_I\left(1+\frac{|x-\xi_i|}{\xi_{i+1}-\xi_i}\right)^{(k-1)p}\left|\theta_i(x)-r_i(x)\right|^p dx$$

$$\int_I\left|\left[L_n(f,x)-P_n(f,x)\right]e^{-\alpha x}\right|^p dx\le c\sum_{i=1}^{n-1}\max_{\xi_i<x<\xi_{i+1}}\left|(p_{i+1}(f,x)-p_i(f,x))e^{-\alpha x}\right|^p\int_I\left(1+\frac{|x-\xi_i|}{\xi_{i+1}-\xi_i}\right)^{(k-1)p}\left[d_j(x)\right]^{(-r+1)p}dx$$

$$=c\sum_{i=1}^{n-1}\max_{\xi_i<x<\xi_{i+1}}\left|(p_{i+1}(f,x)-p_i(f,x))e^{-\alpha x}\right|^p\int_I\left(1+\frac{|x-\xi_i|}{\xi_{i+1}-\xi_i}\right)^{(k-1)p}\left(1+\frac{|x-\xi_i|}{\xi_{i+1}-\xi_i}\right)^{(-r+1)p}dx$$

$$=c\sum_{i=1}^{n-1}\max_{\xi_i<x<\xi_{i+1}}\left|(p_{i+1}(f,x)-p_i(f,x))e^{-\alpha x}\right|^p\int_I\left(1+\frac{|x-\xi_i|}{\xi_{i+1}-\xi_i}\right)^{kp-rp}dx$$

Now we choose r so that $rp\text{-}kp>2$ and it is readily to seen that $\int_I\left[d_i(x)\right]^{-2}dx\le c(\xi_{i+1}-\xi_i)$ $\quad$ $i=1,\dots,n\text{ -}1$ …(5.3)

To prove that

$$\int_{-1}^{1}\left[d_i(x)\right]^{-2}dx=\int_{-1}^{1}\left[1+\frac{|x-\xi_i|}{\xi_{i+1}-\xi_i}\right]^{-2}dx$$

$$|x-\xi_i|=\begin{cases}(x-\xi_i) & \text{if } x>\xi_i\\ -(x-\xi_i) & \text{if } x<\xi_i\end{cases}$$

If $|x-\xi_i|=x-\xi_i$ we take

$$\int_{-1}^{1}\left[d_i(x)\right]^{-2}dx=\int_{-1}^{1}\left[1+\frac{x-\xi_i}{\xi_{i+1}-\xi_i}\right]^{-2}dx=\int_{-1}^{1}\left[\frac{\xi_{i+1}-\xi_i+x-\xi_i}{\xi_{i+1}-\xi_i}\right]^{-2}dx$$

$$=\text{-}(\xi_{i+1}-\xi_i)\left[\frac{\xi_{i+1}+x-2\xi_i}{\xi_{i+1}-\xi_i}\right]^{-1}\Bigg|_{-1}^{1}$$

$$=\text{-}(\xi_{i+1}-\xi_i)\left(\left[\frac{\xi_{i+1}+1-2\xi_i}{\xi_{i+1}-\xi_i}\right]^{-1}-\left[\frac{\xi_{i+1}+1-2\xi_i}{\xi_{i+1}-\xi_i}\right]^{-1}\right)$$

$$=-(\xi_{i+1}-\xi_i)\left(\frac{(\xi_{i+1}-\xi_i)(\xi_{i+1}-1-2\xi_i)-(\xi_{i+1}-\xi_i)(\xi_{i+1}+1-2\xi_i)}{(\xi_{i+1}+1-2\xi_i)(\xi_{i+1}-1-2\xi_i)}\right)$$

$$=-(\xi_{i+1}-\xi_i)\left(\frac{(\xi_{i+1}-\xi_i)(\xi_{i+1}-1-2\xi_i-\xi_{i+1}-1+2\xi_i)}{(\xi_{i+1}+1-2\xi_i)(\xi_{i+1}-1-2\xi_i)}\right)$$

$$=-(\xi_{i+1}-\xi_i)\left(\frac{-2(\xi_{i+1}-\xi_i)}{(\xi_{i+1}+1-2\xi_i)(\xi_{i+1}-1-2\xi_i)}\right)$$

$$=\frac{2(\xi_{i+1}-\xi_i)^2}{(\xi_{i+1}+1-2\xi_i)(\xi_{i+1}-1-2\xi_i)}$$

$$\le C\,(\xi_{i+1}-\xi_i)$$

In the similar way we get the same result take $|x-\xi_i|=-(x-\xi_i)$ .If we applied this result we obtain,

$$\|L_n(f)-P_n\|_{p,\alpha}\le c\sum_{i=1}^{n-1}\max_{\xi_i<x<\xi_{i+1}}\left|(p_{i+1}(f,x)-p_i(f,x))e^{-\alpha x}\right|^p|\xi_{j+1}-\xi_j| \qquad \text{...(5.4)}$$

$$\le c\sum_{i=1}^{n-1}\frac{c}{\xi_{j+1}-\xi_j}\left\|(p_{i+1}(f,x)-p_i(f,x))e^{-\alpha x}\right\|^p_{L_{p,\alpha}[\xi_j,\xi_{j+1}]}|\xi_{j+1}-\xi_j|$$

$$=c\sum_{i=1}^{n-1}\left\|(p_{i+1}(f,x)-p_i(f,x))e^{-\alpha x}\right\|^p_{L_{p,\alpha}[\xi_j,\xi_{j+1}]}$$

$$=c\sum_{i=1}^{n-1}\int_{\xi_j}^{\xi_{j+1}}\left|(p_{i+1}(f,x)-p_i(f,x))e^{-\alpha x}\right|^p dx$$

$$\le c\sum_{i=1}^{n}\int_{\xi_{j-1}}^{\xi_{j+1}}\left|(f(x)-p_i(f,x))e^{-\alpha x}\right|^p dx$$

$$\le c\,\omega_k^{\varphi}(f,\frac{1}{n})^p_{p,\alpha}$$

Proof of theorem (5.2) (part 2) (monotone case)

Here we use the continuous piecewise linear function S^*.For k=2 we can take $p_j(f)$ in section 5 to be the polynomial of degree one and denote $\ell_j^*:=a_jx+b_j$ of theorem (3.2) such that ℓ_j^* is a near-best $L_{p,\alpha}$ approximant for I_k^*.

Definition (5.1): [3]

$$L_n^*(f,x)=p_1(f,-1)+\sum_{j=0}^{n-1}[a_{j+1}(\varphi_j(x)-\varphi_{j+1}(x)) \qquad \text{…(5.5)}$$

$$P_n^*(f,x)=p_1(f,-1)+\sum_{j=0}^{n-1}[a_{j+1}(R_j(x)-R_{j+1}(x)) \qquad \text{…(5.6)}$$

Now we note that since $R_j(x)-R_{j+1}(x)$ is increasing for j=1,…,n-1 and $a_j\ge 0$ for j=1,…,n , the polynomial $P_n^*(f,x)$ is nondecreasing in [-1,1].

Lemma (5.4): [3]

$$L_n^*(f,x)=L_n(f,x) \text{ for } j=1,\dots,n-1$$

Proof of theorem (5.2)

$$\|f(x)-P_n^*(f,x)\|_{p,\alpha}^p=\int_{-1}^{1}\left|L_n^*(f,x)-P_n^*(f,x)\right|^p dx$$

$$=\int_{-1}^{1}\left|\left[f(x)-L_n(f,x)+L_n^*(f,x)-P_n^*(f,x)\right]e^{-\alpha x}\right|^p dx$$

$$\le\int_{-1}^{1}\left|\left[f(x)-L_n(f,x)\right]e^{-\alpha x}\right|^p dx+\int_{-1}^{1}\left|\left[L_n^*(f,x)-P_n^*(f,x)\right]e^{-\alpha x}\right|^p dx$$

In view of theorem (5.1), we have only to estimate the second term,

$$\left|\left[L_n^*(f,x)-P_n^*(f,x)\right]e^{-\alpha x}\right|=\left|\sum_{i=1}^{n-1}(a_{j+1}-a_j)(\varphi_j(x)-R_j(x))e^{-\alpha x}\right|$$

$$\le\sum_{i=1}^{n-1}\left|(a_{j+1}-a_j)(\varphi_j(x)-R_j(x))e^{-\alpha x}\right|$$

$$\le\sum_{i=1}^{n-1}\left|(a_{j+1}-a_j)e^{-\alpha x}\right|\left|\varphi_j(x)-R_j(x)\right|$$

$$\le\sum_{i=1}^{n-1}\frac{\left|\left(p_{i+1}(f,\xi_{i+1})-p_i(f,\xi_{i+1})\right)e^{-\alpha x}\right|}{\xi_{i+1}-\xi_i}\left|\varphi_j(x)-R_j(x)\right|$$

$$=\sum_{i=1}^{n-1}\left|\left(p_{i+1}(f,\xi_{i+1})-p_i(f,\xi_{i+1})\right)e^{-\alpha x}\right|(\xi_{i+1}-\xi_i)^{-1}\left|\varphi_j(x)-R_j(x)\right|$$

Hence by lemma (4.2) and (4.3) we have

$$\int_{-1}^{1}\left|\left[L_n^*(f,x)-P_n^*(f,x)\right]e^{-\alpha x}\right|^p\le\int_{-1}^{1}\sum_{i=1}^{n-1}\left|\left(p_{i+1}(f,\xi_{i+1})-p_i(f,\xi_{i+1})\right)e^{-\alpha x}\right|^p(\xi_{i+1}-\xi_i)^{-p}\left|\varphi_i(x)-R_i(x)\right|^p dx$$

$$\le\int_{-1}^{1}\sum_{i=1}^{n-1}\left(1+\frac{|x-\xi_i|}{\xi_{i+1}-\xi_i}\right)^p\max_{\xi_i<x<\xi_{i+1}}\left|\left(p_{i+1}(f,\xi_{i+1})-p_i(f,\xi_{i+1})\right)e^{-\alpha x}\right|^p(\xi_{i+1}-\xi_i)^{-p}\left|\varphi_i(x)-R_i(x)\right|^p dx$$

$$\le\int_{-1}^{1}\sum_{i=1}^{n-1}\left(1+\frac{|x-\xi_i|}{\xi_{i+1}-\xi_i}\right)^p\max_{\xi_i<x<\xi_{i+1}}\left|\left(p_{i+1}(f,\xi_{i+1})-p_i(f,\xi_{i+1})\right)e^{-\alpha x}\right|^p(\xi_{i+1}-\xi_i)^{-p}(\xi_{i+1}-\xi_i)^{p}\left[d_i(x)\right]^{(-r+2)p}dx$$

$$\le\int_{-1}^{1}\sum_{i=1}^{n-1}\max_{\xi_i<x<\xi_{i+1}}\left|\left(p_{i+1}(f,\xi_{i+1})-p_i(f,\xi_{i+1})\right)e^{-\alpha x}\right|^p\left(1+\frac{|x-\xi_i|}{\xi_{i+1}-\xi_i}\right)^{-(r+2)p}dx$$

Then from the derivation of (5.4) and provided rp-2p>2 we obtain,

$$\int_{-1}^{1}\left|\left[L_n^*(f,x)-P_n^*(f,x)\right]e^{-\alpha x}\right|^p\le c\,\omega_2^{\varphi}(f,\frac{1}{n})_{p,\alpha}^p$$

Then

$$\left\|L_n^*(f,x)-P_n^*(f,x)\right\|_{p,\alpha}^p \le c\,\omega_2^{\varphi}(f,\frac{1}{n})_{p,\alpha}^p$$

$$\left\|f(x)-P_n^*(f,x)\right\|_{p,\alpha}^p \le c\,\omega_k^{\varphi}(f,\frac{1}{n})_{p,\alpha}^p$$

References

1. Husain, L.A., (2010), Unbounded Function Approximation in some $L_{p,\alpha}$ Spaces, M.Sc Thesis, Al-Mustansirya University, Department of Mathematics, College of Science.

2. Hansen, B.M, (2010), Unbounded Functions, M.Sc Thesis, Al-Mustansirya University, Departmen Mathematics, College of Education.

3. Devore, R.A., Leviatan, D. and X. M. Yu, (1992), Polynomial Approximation in L_p $(0 < p < 1)$, Springer-verlage, New York.

4. Devore, R.A. and X.M.Yu, (1985), Poitwise Estimates for Monotone Polynomial Approximation .Constr. Approx, pp.323-331.

5. Leviatan, D. and X.M.Yu, (1991), Shape Preserving Approximation by Polynomial in L_p .

6. Sendov, B. and Popov, V.A, (1981), Average of Modulus of Smoothness of Continuity.

تقريب الدوال في الفضاء $L_{p,\alpha}(I)$ ، $(0 < p < 1)$

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الخلاصة

في هذا البحث وضحنا ان الدالة الغير مقيدة في فضاء الوزن $L_{p,\alpha}$ يمكن ان تقترب من متعددة الحدود الجبرية مع وجود خطأ لا يتجاوز مقاسات التعومة $\omega_k^{\varphi}(f,t,\frac{1}{n})_{p,\alpha}$ للدوال الغير مقيدة في الفضاء $L_{p,\alpha}(I)$.

الكلمات المفتاحية : التقريب الرتيب ، متعددات الحدود، درجة التقريب.