Classification and Construction of (k,3)-Arcs on Projective Plane Over Galois Field GF(7)

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Abstract

The purpose of this work is to study the classification and construction of (k,3)-arcs in the projective plane PG(2,7). We found that there are two (5,3)-arcs, four (6,3)-arcs, six (7,3)-arcs, six (8,3)-arcs, seven (9,3)-arcs, six (10,3)-arcs and six (11,3)-arcs.

All of these arcs are incomplete.

The number of distinct (12,3)-arcs are six, two of them are complete.

There are four distinct (13,3)-arcs, two of them are complete and one (14,3)-arc which is incomplete.

There exists one complete (15,3)-arc.

Key words : Arcs,Projective plane, Galois field

Introduction

A projective plane PG(2,P) over GF(P), P is a prime number consists of $1 + P + P^2$ points and $1 + P + P^2$ lines, 1 + P points on every line and 1 + P lines through every point. Any point of the plane has the form of a triple (x_0,x_1,x_2), where x_0,x_1,x_2 are elements in GF(P) with the exception of a triple consisting of three zero elements. Two triples (x_0,x_1,x_2), (y_0,y_1,y_2) represent the same point if there exists λ in GF(P) \ {0} such that $(y_0,y_1,y_2) = \lambda (x_0,x_1,x_2)$.

Similarly, any line of the plane has the form of a triple $[x_0,x_1,x_2]$, where x_0,x_1,x_2 are in GF(P) with the exception of a triple consisting three zero elemens. Two triples $[x_0,x_1,x_2]$, $[y_0,y_1,y_2]$ represent the same line if there exists λ in GF(P) \ {0} such that $[y_0,y_1,y_2] = \lambda [x_0,x_1,x_2]$.

A point $P(x_0,x_1,x_2)$ is incident with the line $[y_0,y_1,y_2]$ iff:

$$x_0y_0 + x_1y_1 + x_2y_2 = 0$$

Definition 1: [1,2]

A (k,3)-arc in PG(2,P) is a set of k points no four of them are collinear.

Definition 2: [1,2]

A (k,3)-arc is complete if it is not contained in a (k + 1,3)-arc.

Definition 3: [3]

The i-secant of a (k,3)-arc is a line intersects the arc in exactly i points, for a (k,3)-arc, each line of PG(2,P) is a 3-secant, 2-secant, 1-secant, or 0-secant.

A 3-secant is called a trisecant.

A (k,3)-arc is complete if every point of PG(2,P) lies on some trisecant of the arc.

Let r_i be the number of the i-secants of a (k,3)-arc in the plane which are r_3, r_2, r_1, r_0.

Definition 4: [1,2]:

A point N not on a (k,3)-arc has index i, denoted by N_i, if there are exactly i-trisecants of the arc through N.

Let $C_i = |N_i|$ be the number of the points N_i.

Thus, a (k,3)-arc is complete iff $C_0 = 0$.

Lemma 1: [4]

Let r_i be the total number of the i-secants of a (k,n)-arc in PG(2,P), then the following equations are hold:

$$\sum_{i=0}^{n} r_i = q^2 + q + 1$$

$$\sum_{i=1}^{n} i r_i = k(q+1)$$

$$\sum_{i=2}^{n} i(i-1) r_i = k(k-1)$$

Definition 5: [4]

Let r_i be the total number of i-secants of a (k,n)-arc in PG(2,P), then the type of a (k,n)-arc w.r.t. its lines is denoted by $(r_n,r_{n-1},\ldots,r_0)$.

Lemma 2: [4]

Let (k_1,n)-arc be of type $(r_n,r_{n-1},\ldots,r_0)$ and a (k_2,n)-arc be of type $(t_n,t_{n-1},\ldots,t_0)$, then (k_1,n) and (k_2,n) have the same type iff $r_i = t_i$ for all i.

Definition 6: [2]

Two arcs are projectively equivalent under type of lines iff they have the same type.

Definition 7: [5]

Let Q_1, Q_2 be two points in PG(2,P) which are not on a (k,n)-arc and let $K_1 = K\cup\{Q_1\}$, $K_2 = K\cup\{Q_2\}$, then Q_1, Q_2 are in the same set iff (k_1,n) and (k_2,n) are projectively equivalent under type of lines.

Lemma 3: [5]

Let Q_1, Q_2 be two points in PG(2,P) which are not on a (k,n)-arc, then

(i) Q_1, Q_2 are in the same set if they have the same type.

(ii) Q_1, Q_2 are in the different sets if they have the different types.'

Let P_i and ℓ_i, i = 1, 2, …, 57 be the points and lines of PG(2,7), respectively.
Let i stands for the points P_i, all the points and the lines of PG(2,7) are given in the table.

Definition 8: [2]

A projectively of (k,3)-arc is a non-singular (3×3) matrix which keeps the k-points of the arc point wise or globally invariant.

The Construction of the Projectively Distinct (5,3)-arcs in PG(2,7):

Let A = {1,2,9,17} be a set of reference points in PG(2,7) no three of them are collinear. The distinct (5,3)-arcs can be constructed by adding to A in each time a point from the remaining 53 points of the plane. By definition 6, there are only two projectively distinct (5,3)-arcs, which are:

$B_1 = \{1,2,9,17,3\}$ and $B_2 = \{1,2,9,17,4\}$.

The set of projectivities fixing a (k,3)-arc B forms the group G(B).

The group $G(B_1)$ has eight projectivities:

$$T_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ T_2=\begin{bmatrix}6&0&0\\0&6&0\\1&1&1\end{bmatrix},\ T_3=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\ T_4=\begin{bmatrix}0&6&0\\6&0&0\\1&1&1\end{bmatrix},\ T_5=\begin{bmatrix}0&0&6\\1&1&1\\6&0&0\end{bmatrix},$$

$$T_6=\begin{bmatrix}0&0&6\\1&1&1\\0&6&0\end{bmatrix},\ T_7=\begin{bmatrix}1&1&1\\0&0&6\\6&0&0\end{bmatrix},\ T_8=\begin{bmatrix}1&1&1\\0&0&6\\0&6&0\end{bmatrix}$$

So $G(B_1) \cong D_4$, while $G(B_2)$ has four projectivities:

$$T_1=I\ ,\ T_2=\begin{bmatrix}6&0&0\\0&6&0\\1&1&1\end{bmatrix},\ T_3=\begin{bmatrix}6&0&0\\2&1&0\\0&0&1\end{bmatrix},\ T_4=\begin{bmatrix}1&0&0\\5&6&0\\1&1&1\end{bmatrix}$$

$G(B_2) \cong Z_2\times Z_2$.

The Classification and Construction of (6,3)-Arcs:

There are 42 points of index zero for B_1. Then $G(B_1)$ partitions these points into 9 orbits.

So, we have nine (6,3)-arc to be constructed by adding one point from each of these nine orbits to B_1.

We have 47 points of index zero for B_2. $G(B_2)$ partitions these points into 15 orbits. So we have 15 (6,3)-arcs to be constructed by adding one point from each of these orbits to B_2. By definition 6, we have four (6,3)-arcs which are projectivity distinct :

$C_1 = \{1,2,9,17,3,10\}$, $C_2 = \{1,2,9,17,3,11\}$, $C_3 = \{1,2,9,17,3,26\}$, $C_4 = \{1,2,9,17,4,28\}$.

$G(C_1) \cong S_4$, $G(C_2) \cong Z_3$, $G(C_3) \cong Z_2$, $G(C_4) \cong I$.

The Classification and Construction of (7,3)-Arcs:

The groups $G(C_1)$, $G(C_2)$, $G(C_3)$ and $G(C_4)$ partition the points of index zero for C_1, C_2, C_3 and C_4 into 4, 12, 24, 46 orbits, respectively. By definition 6, we have only six (7,3)-arcs which are projectively distinct. These are:

$D_1 = \{1,2,9,17,3,10,16\}$, $D_2 = \{1,2,9,17,3,11,16\}$, $D_3 = \{1,2,9,17,3,11,20\}$,
$D_4 = \{1,2,9,17,3,11,26\}$, $D_5 = \{1,2,9,17,3,26,32\}$, $D_6 = \{1,2,9,17,4,28,50\}$.

The groups $G(D_2)$, $G(D_3)$, $G(D_4)$ and $G(D_6)$ are isomorphic to the identity group, $G(D_1) \cong S_4$ and $G(D_5) \cong Z_3$.

We have six distinct (7,3)-arcs, each one of them is incomplete.

The Classification and Construction of (8,3)-Arcs:

The groups $G(D_1)$, $G(D_2)$, …, $G(D_6)$ partition the points of index zero for D_1, D_2, …, D_6 into 2, 25, 30, 33, 20, 45 orbits, respectively. By definition 6, we have only six (8,3)-arcs which are projectively distinct:

$E_1 = \{1,2,9,17,3,10,16,26\}$, $E_2 = \{1,2,9,17,3,10,16,27\}$, $E_3 = \{1,2,9,17,3,11,16,26\}$,
$E_4 = \{1,2,9,17,3,11,26,20\}$, $E_5 = \{1,2,9,17,3,26,32,22\}$, $E_6 = \{1,2,9,17,4,28,50,31\}$.

The groups $G(E_3)$, $G(E_4)$, $G(E_5)$ and $G(E_6)$ are isomorphic to the identity group, $G(E_1) \cong Z_2$ and $G(E_2) \cong Z_3$, we have six distinct (8,3)-arcs, each one of them is incomplete.

The Classification and Construction of (9,3)-Arcs:

The groups $G(E_1)$, $G(E_2)$, …, $G(E_6)$ partition the points of index zero for E_1, E_2, …, E_6 into 8, 7, 24, 29, 34, 39 orbits, respectively. By definition 6, we have only seven (9,3)-arcs which are projectively distinct, these are:

$F_1 = \{1,2,9,17,3,10,16,26,29\}$, $F_2 = \{1,2,9,17,3,10,16,26,27\}$, $F_3 = \{1,2,9,17,3,10,16,27,28\}$,
$F_4 = \{1,2,9,17,3,10,16,27,39\}$, $F_5 = \{1,2,9,17,3,11,26,20,47\}$, $F_6 = \{1,2,9,17,4,28,50,31,12\}$,
$F_7 = \{1,2,9,17,4,28,50,31,42\}$

The groups $G(F_3)$, $G(F_4)$, $G(F_5)$, $G(F_6)$ and $G(E_7)$ are isomorphic to the identity group, $G(F_1) \cong Z_3$ and $G(F_2) \cong Z_2$.

The Classification and Construction of (10,3)-Arcs:

The groups $G(F_1)$, $G(F_2)$, …, $G(F_7)$ partition the points of index zero for F_1, F_2, …, F_7 into 5, 13, 8, 18, 26, 30, 33 orbits, respectively. By definition 6, we have only six (10,3)-arcs which are projectively distinct, these arcs are:

$H_1 = \{1,2,9,17,3,10,16,26,29,32\}$, $H_2 = \{1,2,9,17,3,10,16,26,27,32\}$,
$H_3 = \{1,2,9,17,3,10,16,26,29,39\}$, $H_4 = \{1,2,9,17,3,10,16,27,28,39\}$,
$H_5 = \{1,2,9,17,4,28,31,50,12,42\}$, $H_6 = \{1,2,9,17,4,28,31,50,12,42\}$.

The groups $G(H_1)$, $G(H_2)$, $G(H_3)$, $G(H_4)$, $G(H_5)$ and $G(E_6)$ are isomorphic to the identity group. We have only six distinct (10,3)-arcs, each one of them is in complete.

The Classification and Construction of (11,3)-Arcs:

The groups $G(H_1)$, …, $G(H_6)$ partition the points of index zero for H_1, …, H_6 into 5, 8, 11, 14, 20, 22 orbits. By definition 6, we have only six (11,3)-arcs which are projectively distinct, these arcs are:

$I_1 = \{1,2,9,17,3,10,16,26,29,32,36\}$, $I_2 = \{1,2,9,17,3,10,16,26,27,32,36\}$,
$I_3 = \{1,2,9,17,3,10,16,26,27,32,39\}$, $I_4 = \{1,2,9,17,3,10,27,28,39,36\}$,
$I_5 = \{1,2,9,17,3,10,16,27,28,39,54\}$, $I_6 = \{1,2,9,17,4,28,31,50,12,42,25\}$.

The groups $G(I_i)$, $i = 1,2,\ldots,6$ are isomorphic to the identity group.

The Classification and Construction of (12,3)-Arcs:

The groups $G(I_1)$, …, $G(I_6)$ partition the points of index zero for I_1, …, I_6 into 2, 4, 4, 10, 10, 16 orbits, respectively. By definition 6, we have only six (12,3)-arcs which are projectively distinct. These arcs are:

L_1 = {1,2,9,17,3,10,16,26,29,32,36,54}, L_2 = {1,2,9,17,3,10,16,26,29,32,36,46},
L_3 = {1,2,9,17,3,10,16,26,27,32,36,46}, L_4 = {1,2,9,17,3,10,16,27,28,39,36,43},
L_5 = {1,2,9,17,3,10,16,27,28,39,36,46}, L_6 = {1,2,9,17,4,28,31,50,12,42,25,55}.

The groups $G(L_i)$, i = 1,2,…,6 are isomorphic to the identity group. We have six distinct arcs, two of them are complete and the others are incomplete.

The Classification and Construction of (13,3)-Arcs:

The groups $G(L_1)$, …, $G(L_6)$ partition the points of index zero for L_1, …, L_6 into 1, 4, 5, 4, 5, 15 orbits, respectively. By definition 6, we have only four (13,3)-arcs which are projectively distinct:

M_1 = {1,2,9,17,3,10,16,26,27,32,36,46,43}, M_2 = {1,2,9,17,3,10,16,27,28,39,36,43,34},
M_3 = {1,2,9,17,3,10,16,27,28,39,36,46,43}, M_4 = {1,2,9,17,4,28,31,50,12,42,25,55,18}.

The groups $G(M_1)$, $G(M_2)$ and $G(M_4)$ are isomorphic to the identity group, $G(M_3) \cong Z_2$. M_1 and M_2 are complete arcs, while M_3 and M_4 are incomplete arcs.

The Classification and Construction of (14,3)-Arcs:

The groups $G(M_3)$ and $G(M_4)$ partition the points of index zero for M_3 and M_4 into 1 and 2 orbits, respectively. So there exist three (14,3)-arcs to be constructed. By definition 6, we have only one (14,3)-arcs which is incomplete:

N = {1,2,9,17,3,10,16,27,28,39,36,46,43,54}.

G(N) is isomprphic to the identity group.

The Classification and Construction of (15,3)-Arcs:

The point 55 is the only point of index zero for N.

We construct (15,3)-arc by adding the point 55 to N, then:

Q = {1,2,9,17,3,10,16,27,28,39,36,46,43,54,55}

is a complete (15,3)-arc. G(Q) is isomorphic to the identity group.

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Table 1 : Points and Lines of PG(2,7)

i	P_i			ℓ_i							
1	1	0	0	2	9	16	23	30	37	44	51
2	0	1	0	1	9	10	11	12	13	14	15
3	1	1	0	8	9	22	28	34	40	46	52
4	2	1	0	5	9	19	29	32	42	45	55
5	3	1	0	4	9	18	27	36	38	47	56
6	4	1	0	7	9	21	26	31	43	48	53
7	5	1	0	6	9	20	24	35	39	50	54
8	6	1	0	3	9	17	25	33	41	49	57
9	0	0	1	1	2	3	4	5	6	7	8
10	1	0	1	2	15	22	29	36	43	50	57
11	2	0	1	2	12	19	26	33	40	47	54
12	3	0	1	2	11	18	25	32	39	46	53
13	4	0	1	2	14	21	28	35	42	49	56
14	5	0	1	2	13	20	27	34	41	48	55
15	6	0	1	2	10	17	24	31	38	45	52
16	0	1	1	1	51	52	53	54	55	56	57
17	1	1	1	8	15	21	27	33	39	45	51
18	2	1	1	5	12	22	25	35	38	48	51
19	3	1	1	4	11	20	29	31	40	49	51
20	4	1	1	7	14	19	24	36	41	46	51
21	5	1	1	6	13	17	28	32	43	47	51
22	6	1	1	3	10	18	26	34	42	50	51
23	0	2	1	1	30	31	32	33	34	35	36
24	1	2	1	7	15	20	25	30	42	47	52
25	2	2	1	8	12	18	24	30	43	49	55
26	3	2	1	6	11	22	26	30	41	45	56
27	4	2	1	5	14	17	27	30	40	50	53
28	5	2	1	3	13	21	29	30	38	46	54
29	6	2	1	4	10	19	28	30	39	48	57
30	0	3	1	1	23	24	25	26	27	28	29
31	1	3	1	6	15	19	23	34	38	49	53
32	2	3	1	4	12	21	23	32	41	50	52
33	3	3	1	8	11	17	23	36	42	48	54
34	4	3	1	3	14	22	23	31	39	47	55
35	5	3	1	7	13	18	23	35	40	45	57
36	6	3	1	5	10	20	23	33	43	46	56
37	0	4	1	1	44	45	46	47	48	49	50
38	1	4	1	5	15	18	28	31	41	44	54
39	2	4	1	7	12	17	29	34	39	44	56
40	3	4	1	3	11	19	27	35	43	44	52
41	4	4	1	8	14	20	26	32	38	44	57
42	5	4	1	4	13	22	24	33	42	44	52
43	6	4	1	6	10	21	25	36	40	44	55
44	0	5	1	1	37	38	39	40	41	42	43
45	1	5	1	4	15	17	26	35	37	46	55
46	2	5	1	3	12	20	28	36	37	45	53
47	3	5	1	5	11	21	24	34	37	47	57
48	4	5	1	6	14	18	29	33	37	48	52
49	5	5	1	8	13	19	25	31	37	50	56
50	6	5	1	7	10	22	27	32	37	49	54
51	0	6	1	1	16	17	18	19	20	21	22
52	1	6	1	3	15	16	24	32	40	48	56
53	2	6	1	6	12	16	27	31	42	46	57
54	3	6	1	7	11	16	28	33	38	50	55
55	4	6	1	4	14	16	25	34	43	45	54
56	5	6	1	5	13	16	26	36	39	49	52
57	6	6	1	8	10	16	29	35	41	47	53

تصنيف وبناء الاقواس (k,3) في مستوي إسقاطي حول حقل كالوا GF(7)

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الخلاصة

الغرض من هذا البحث هو لدراسة تصنيف وبناء الاقواس (k,3) في المستوي الاسقاطي G(2,7) . لقد وجدنا قوسين – (5,3)، اربعة اقواس – (6,3)، ستة أقواس – (7,3)، ستة أقواس – (8,3) سبعة أقواس – (9,3)، ستة أقواس – (10,3)، وستة أقواس – (11,3)، كل هذه الاقواس تكون غير كاملة.
عدد الاقواس المختلفة – (12,3) تكون ستة، اثنان منها تكون كاملة. توجد أربعة أقواس – (13,3) اثنان منها كاملة وقوس واحد – (14,3) غير كامل.
يوجد قوس واحد – (15,3) يكون كاملاً.

الكلمات المفتاحية : أقواس ، مستوي اسقاطي ، حقل كالوا