Solving Fuzzy-Parametric Linear Programming Problems

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Received in: 19 June 2012 , Accepted in: 12 September 2012

Abstract
The fuzzy sets theory has been applied in many fields, such as operations research, control theory and management sciences, etc. In particular, an application of this theory in decision making problem is linear programming problems with fuzzy technological coefficients numbers, as well as studying the parametric linear programming problems in the case of changes in the objective function. In this paper presenting a new procedure which connects and makes link between fuzzy linear programming problem with fuzzy technological coefficients numbers and parametric linear programming problem with change in coefficients of the objective function, then develop a numerical example illustrates the steps of solution to this kind of problems.

Keywords: Fuzzy linear programming; parametric linear programming; fuzzy- parametric linear programming.

Introduction

The linear programming is one of the important tools to help decision- maker to take the best decision, but it needs to a certain date (exact) and this is not realistic because the environmental conditions often provide us data which are uncertain. Therefore, the concept of fuzzy set theory was used to solve the linear programming problems when uncertain date by Bellman and Zadeh[1]. Tanaka [2] hired this concept to solve the fuzzy linear programming with multiple goals. Followed by Nagoita[3] formulated the fuzzy linear programming problems.

The issue of parametric linear programming problems studied by Saaty and Gass[4] after determining the feasible solution and the values of parameters, its could be possible to generate all solutions with keeping to optimal solution.

In our paper representing idea (proposed) to make link between fuzzy linear programming and parametric linear programming in the case of one vehicle.

We consider in fuzzy linear programming the technological coefficients are fuzzy numbers, but in parametric linear programming the change will be happened in coefficients of objective function.

The paper is outlined as follows: Introduction in section1. Linear programming problem in section 2. Fuzzy linear programming problems with technological coefficients are fuzzy number in section 3. In section 4 we studied the parametric linear programming problems with change in coefficients of objective function. But in section 5 we studied a new proposed which call it fuzzy- parametric linear programming problems. After that in section 6 we examined the application of fuzzy- parametric linear programming to concrete example.

I) Linear Programming Problems [5]**:**

Let $X= (x_1, x_2, \dots, x_n)$ be (n) variables and (m) constraints, then the linear programming problems is defined as follows:

$$\text{Maximize (minimize) } Z= \left\{ CX \middle| \sum_{j=1}^{n} P_j x_j \le (\ge) b, X \ge 0 \right\} \ldots(1)$$

Where:

P_j: The original basic columns in the matrix **A**, where $P_h= a_{ih}$, i= 1, 2,…, m and h $\in\{1,2,\dots n\}$

x_j: Decision variables.

b: Right hand side values.

II) Fuzzy Linear Programming Problems:

If any coefficient for linear programming problems is uncertain value, then it is called fuzzy linear programming problems and its formis defined as:

$$\text{Maximize (minimize) } Z= \left\{ CX \middle| \sum_{j=1}^{n} \widetilde{P}_j x_j \le (\ge) b, X \ge 0 \right\} \ldots(2)$$

Where:

$\widetilde{P}_j$: The original fuzzy columns in the matrix A

Assumption 1: By using fuzzy logic which defined by L.A. Zadeh[6] the membership function of fuzzy technological coefficients are presented as:

$$M_{\underset{A}{\sim}}(x) = \begin{cases} 1 & \ldots\text{if } x \le A \\ (A + A^t - x)/A^t & \ldots\text{if } A < x < A + A^t \\ o & \ldots\text{if } A + A^t \le x \end{cases} \ldots(3)$$

Where: $x \in R$

A^t: The tolerance values of each element in the matrix A where A= a_{ij} and $A^t = d_{ij} > 0$.
For defuzzification of the problem (2), we first fuzzify the objective function, by retailing it into two subproblems (Z_1, Z_2) as follows (only maximize):

$$\text{Maximize } Z_1 = \left\{ CX \middle| \sum_{j=1}^{n} P_j x_j \le b, X \ge 0 \right\} \qquad \text{...(4)}$$

And

$$\text{Maximize } Z_2 = \left\{ CX \middle| \sum_{j=1}^{n} (P_j + P_j^t) x_j \le b, X \ge 0 \right\} \qquad \text{...(5)}$$

P_j^t : The tolerance columns of original columns (matrix $\mathbf{A^t}$).

Solving these two subproblems by using regular simplex method can obtain two different values Z_1 and Z_2 [7].
Then we can determine the upper and lower bound as Z_u and Z_ℓ.
Where:
Z_u= max. { Z_1, Z_2} … (6)
And, $Z\ell$ = min. { Z_1, Z_2} … (7)

The objective function takes values between $Z\ell$, Z_u while technological coefficients vary between (a_{ij}) and (a_{ij}+d_{ij}). In more certain the objective function takes the maximum value Z_u when the technological coefficients at the lower value (a_{ij}), and the minimum value Z_ℓ at the upper value (a_{ij}+d_{ij}).
This means the membership function of the objective function G which is subset in R^n is defined as [7]:

$$M_G(CX) = \begin{cases} 0 & \text{...if } CX \le Z_\ell \\ (CX - Z_\ell)/(Z_u - Z_\ell) & \text{...if } Z_\ell < CX < Z_u \\ 1 & \text{...if } Z_u \le CX \end{cases} \qquad \text{...(8)}$$

And it illustrated in fig. (1).

The fuzzy set of the i-th constraints C_i which is a subset in R^m, is defined by

$$M_{Ci}(X) = \begin{cases} 0 & \text{...if } b \le AX \\ (b - AX)/A^t X & \text{...if } AX < b < (A + A^t)X \\ 1 & \text{...if } (A + A^t)X \le b \end{cases} \qquad \text{...(9)}$$

And it illustrate in fig. (2).

By using the definition of the fuzzy decisive proposed by Bellman and Zadeh[1] we have:
$M_D(x)$ = Min {$M_G(x)$, Min ($M_{ci}(x)$)} … (10)
In this case the optimal fuzzy decision is a solution of the problem:

$$\max_{x \ge 0}(M_D(x)) = \max_{x \ge 0} \text{Min } \{M_G(x), \text{Min } (M_{ci}(x))\} \qquad \text{… (11)}$$

Consequently, the problem (3) becomes to the following optimization problem [6]:

$$\left.\begin{array}{l} \text{Max } \lambda \\ M_G(x) \geq \lambda \\ M_{ci}(x) \geq \lambda,\ i=1, 2, \ldots m \\ X \geq 0,\ 0 \leq \lambda \leq 1. \end{array}\right\} \quad \ldots(12)$$

By using (8) and (9), the problem (12) can be written as:

Max λ

$$\left.\begin{array}{l} \lambda(Z_u - Z_\ell) + Z_\ell - CX \leq 0 \\ (A + \lambda A^t)X - b_i \leq 0,\ i = 1, 2, \ldots, m \\ x \geq 0,\ 0 \leq \lambda \leq 1 \end{array}\right\} \quad \ldots(13)$$

$\lambda\ (z_u - z_\ell) + z_\ell$ its a linear function of λ then we called it a slope function and symbol as f(λ). Solving this problem by using bisection method to obtain λ^* which is the optimal value of λ, such that the value of objective function (CX^*) becomes greater than value of $f(\lambda^*)$, by other words the intersection of the objective function at λ^* with the feasible solutions set is non- empty.

The condition stopping of the bisection method is [8],[9]:

$$\left|\lambda_{i+1} - \lambda_i\right| < \varepsilon \quad \ldots.(14)$$

Now, we can write the problem (2) with λ^* as:

$$\text{Maximize (Min.) } Z^* = \left\{ CX^* \left| \sum_{j=1}^{n} P^*_j\, x^*_j \leq (\geq) b, X^* \geq 0 \right. \right\} \quad \ldots(15)$$

Where:

P^*: The original crisp columns, its elements $a_{ij}+\lambda^* d_{ij}$ are approximate.

And:

$$P^*_j = P_j + \lambda^* P_j^t \quad \ldots.(16)$$

X^*, Z^* are symmetric to X, Z but at the λ^* level.

III) Parametric Linear Programming Changes in C [5], [10]**:**

Parametric linear programming is an extension of the sensitivity optimal analysis. It investigates the effect of predetermined continuous variations in the objective function coefficients on the optimum solution.

Let $X = (x_1, x_2, \ldots, x_n)$ and define the parametric linear programming changes in C as:

$$\text{Maximize } Z = \left\{ (C + C't)X \left| \sum_{j=1}^{n} P_j x_j \leq b, X \geq 0 \right. \right\} \quad \ldots.(17)$$

Where:

C': Variation profit vector $C' = (c'_1, c'_2, \ldots., c'_n)$.

Let X_{Bi}, B_i, $C_{Bi}(t)$ be the elements that define the optimal solution associated with critical value $\boldsymbol{t_i}$. Starting at $\boldsymbol{t_0} = 0$ with B_0 as its optimal basis. Next, the critical value $\boldsymbol{t_{i+1}}$ where (i = 0, 1, 2, …) and its optimal basis, if one exists, are determined. The changes in C can affect only the optimality of the problem, the current solution $X_{Bi} = B_i^{-1} b$ will remain optimal $\boldsymbol{t} \geq \boldsymbol{t_i}$ so long the reduced profit, $Z_j(t) - c_j(t)$, satisfies the following optimality condition:

$$Z_j(t) - C_j(t) = C_{Bi}(t)B_i^{-1}P_j - C_j(t) \geq 0, \text{for all j nonbasis column} \quad ....(18)$$

The value of t_{i+1} equals the largest t in $[t_i, t_{i+1}]$ that satisfies all the optimality conditions [5]. There is a function of t represents to the objective function in each interval. Such that [10]:

$$Z(t) = Z + tZ' \quad ...(19)$$

Where:

$$Z = CX \quad ...(20)$$

And

$$Z' = C'X \quad ...(21)$$

The optimal solution for the entire range of t is summarized in table (1) [5]:

To find all critical values which determine the intervals of t with the optimal solution in each alternative basic.

IV) Fuzzy- Parametric Linear Programming Problems:

Now, suggesting new procedure which connect between fuzzy linear programming with technological coefficients are uncertain and parametric linear programming with change in coefficients of objective function which will define it as follows:

$$\text{Maximize } Z = \left\{ (C + C't)X \middle| \sum_{j=1}^{n} \tilde{P}_j x_j \leq b, X \geq 0 \right\} ....(22)$$

The general idea of solving fuzzy- parametric linear programming problems is to start with the optimal solution at **t** = 0. Next, solving by fuzzy linear programming as in (II) to obtain λ^* which make the problem in a crisp. Then, solving by using the optimality condition in (18) which become as:

$$Z^*{}_j(t) - C_j(t) = C_{Bi^*}(t)(B^*{}_i)^{-1}P^*{}_j - C_j(t) \geq 0 \quad ......(23)$$

For all j are non basic columns.

Numerical Example:

This example is found by the research which are as follows:

Max Z = $(3-6t)\, x_1 + (2-2t)\, x_2 + (5+5t)\, x_3$

Subject to:

$$\tilde{1}x_1 + \tilde{2}x_2 + \tilde{1}x_3 \leq 40$$
$$\tilde{3}x_1 + \tilde{2}x_3 \leq 60$$
$$\tilde{1}x_1 + \tilde{4}x_2 \leq 30$$

$x_j \geq 0$, $d_{ij} = 0.1$ for each $a_{ij} \neq 0$ and $d_{ij} = 0$ otherwise.

Solution:

1) Let $t = 0$ then, the problem become without parametric changes, i.e. it has turned into a fuzzy linear programming problem.

2) Retail the problem to two subproblems and solving by simplex method:
$Z_\ell = 151.0204$ and $Z_u = 160$

3) Now, formulate the problem as in (13) we get:

$$\max \lambda$$
$$8.9796\lambda + 151.0204 - (3x_1 + 2x_2 + 5x_3) \le 0$$
$$(1+0.1\lambda)x_1 + (2+0.1\lambda)x_2 + (1+0.1\lambda)x_3 \le 40$$
$$(3+0.1\lambda)x_1 + (2+0.1\lambda)x_3 \le 60$$
$$(1+0.1\lambda)x_1 + (4+0.1\lambda)x_2 \le 30$$
$$x_1, x_2, x_3 \ge 0 \; and \; 0 \le \lambda \le 1$$

To find the values of CX by simplex and the value of f(λ) for each level of λ.
For λ = 0 implice Cx = 160 and f(0) = 151.0204, this means the intersection is *non- empty*.
For λ= 1 implice CX= 151.0204 and f(1) =160, this means the intersection is *empty*.
And by using the bysection method the first point is λ_1 = 0.5 that implice CX=155.3836965 and f(0.5) =155.5102, this means the intersection is *empty*.
Therefore, the next λ will be λ_2= 0.25 *non- empty*.
At the end we obtain λ^* = 0.492954735 implice the optimal solution is $Z_0^* = CX_{B_\circ}{}^* = 155.446945 \;\; is\, non-empty.$

$X_{B_0^*} = (x_2^*, x_3^*, x_6^*)\, implice\, C_{B_0^*}(t) = (2-2t \quad 5+5t \quad 0)$,

$x_2^* = 4.527583, x_3^* = 29.278354 \, and \, x_6^*$ is not real value.

$$(B_0^*)^{-1} = \begin{bmatrix} 0.487973 & -0.249855 & 0 \\ 0 & 0.487973 & 0 \\ -1.975945 & 1.011738 & 1 \end{bmatrix}$$

The crisp original columns become as:

$$P_1^* = P_1 + \lambda^* P_1^t, P_2^* = P_2 + \lambda^* P_2^t, P_3^* = P_3 + \lambda^* P_3^t$$

or:

$$P_1^* = \begin{bmatrix} 1.049295 \\ 3.049295 \\ 1.049295 \end{bmatrix}, P_2^* = \begin{bmatrix} 2.049295 \\ 0 \\ 4.049295 \end{bmatrix}, P_3^* = \begin{bmatrix} 1.049295 \\ 2.049295 \\ 0 \end{bmatrix}$$

We apply the optimality condition in (23):

$C_{B_0^*}(t)(B_0^*)^{-1} = (0.975946 - 0.975946t \quad -0.49971 + 0.49971t \quad 0)$

$C_{B_0^*}(t)(B_0^*)^{-1} P_j^* - C_j(t) \ge 0 \; for \;\; j = 1,4,5$

To find the first critical value t_1=1 this means the first interval is [0, 1]. And to find the alternative basic (B_1^*) as a second basic therefore P_4 must enter and P_2 is leaving the second basic.

$X_{B_1^*} = (x_4^*, x_3^*, x_6^*)\, implice\, C_{B_1^*}(t) = (0 \quad 5+5t \quad 0)$

And $x_3^* = 29.2783546, \quad x_4^*, x_6^* \, are\, not\, real \;\, value.$

$Z_1^* = 5x_3^* = 146.391773$

$$(B_1^*)^{-1} = \begin{bmatrix} 1 & -0.512025 & 0 \\ 0 & 0.487973 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_{B_1^*}(t)(B_1^*)^{-1} = (0 \quad 2.439865 + 2.439865t \quad 0)$$

$$C_{B_1^*}(t)(B_1^*)^{-1}P_j^* - C_j(t) \geq 0 \; for \;\; j = 1,2,5$$

To find the second critical value we apply the optimality condition in (23). But in this case ***t*** goes to infinite i.e. there is no second critical value, and the second interval is $[1,\infty)$ therefore, the parametric analysis ends.

Now, we apply (19), (20), (21) to find z*(***t***):

$$(Z_0^*)' = -2x_2^* + 5x_3^* = 137.336604$$

Then: $Z_0^*(t) = 155.446945 + 137.336604\ t$

And

$$(Z_1^*)' = C'X_{B_1^*} = 146.391773$$

Then: $Z_1^*(t) = 146.391773 + 146.391773t$

The summarized solution illustrate in table (2).

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Table (1) the Summarized Solution

t	X	Z
All intervals	Value of variables in each interval	Value of z in each interval

Table (2) the Summarized Numerical Solution

t	x_1^*	x_2^*	x_3^*	$Z^*(t)$
$0 \le t \le 1$	0	4.527583	29.278354	155.446945+137.336604 t
$1 \le t < \infty$	0	0	29.278354	146.391773+146.391773t

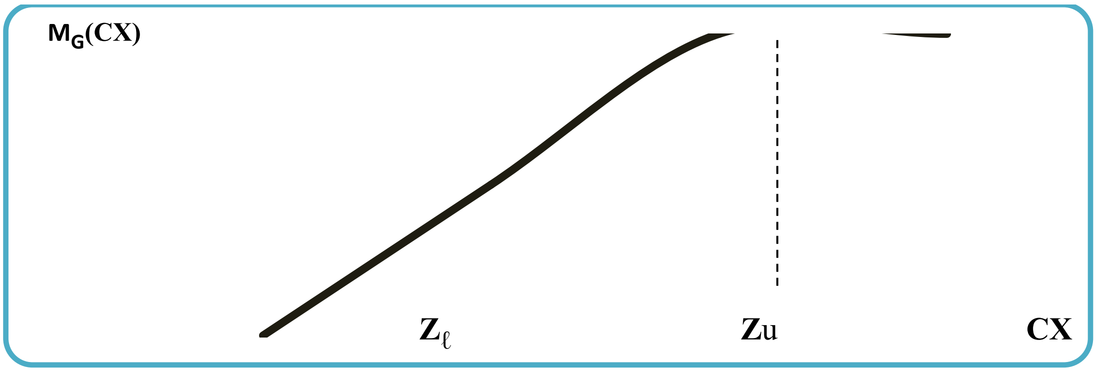

Figure (1) Membership functions of the objective function

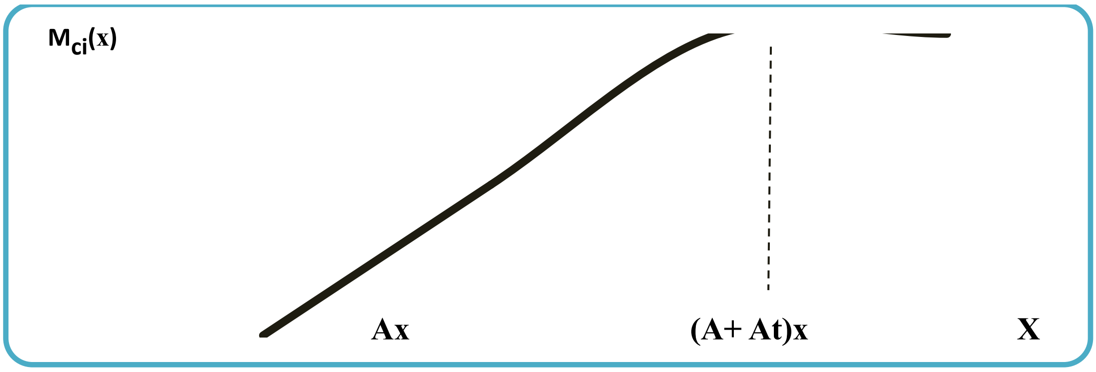

Figure (2) Membership functions of the constraints

حل مشاكل البرمجة الخطية الضبابية – المعلمية

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استلم البحث في:19 حزيران 2012 ، قبل البحث في:12 ايلول 2012

الخلاصة

ان نظرية المجموعات الضبابية يمكن ان تطبق في الكثير من المجالات، على سبيل المثال:بحوث العمليات، نظرية السيطرة ، علم الادارة... الخ.
ان احدى تطبيقات نظرية المجموعات الضبابية هي في مشاكل اتخاذ القرار حيث تم تطبيقها في مشاكل البرمجة الخطية مع اعداد ضبابية للمعاملات التكنولوجية (معاملات اتخاذ القرار) بالاضافة الى دراسة مشاكل البرمجة الخطية المعلمية في حالة التغير في معاملات دالة الهدف.
في هذا البحث سنعرض اسلوب جديد يربط بين مشكلة البرمجة الخطية الضبابية مع معاملات تكنولوجية ضبابية ومشكلة البرمجة الخطية المعلمية مع تغيرات في معاملات دالة الهدف، ومن ثم صياغة مثال عددي يوضح خطوات الحل لهذا النوع من المشاكل.

الكلمات المفتاحية: برمجة خطية ضبابية، برمجة خطية معلمية، برمجة خطية ضبابية-معلمية.