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Degree of Monotone Approximation in $L_{p,\alpha}$ Spaces

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Abstract

The aim of this paper is to study the best approximation of unbounded functions in the weighted spaces $L_{p,\alpha}, p \geq 1, \alpha > 0$.

Key Words: Weighted space, unbounded functions, monotone approximation

Introduction

With a great potential for applications to a wide variety of problems, approximation theory represents on old field of mathematical research. In the fifties a new breath over it has been brought by a systematic study of the linear methods of approximation which are given by sequences of linear operators .These methods became a firmly entrenched part of approximation theory. The problem of function connected with different polynomials was examined in many paper like[1] and [2]. In this paper we studied the degree of approximation of unbound functions by using piecewise monotone polynomials in the weighted spaces $L_{p,\alpha}$.

Definitions and notations

Let f be any function such that $|f(x)| \leq Me^{\alpha x}, \alpha > 0, M \in R, x \in [a,b]$ we denote by $L_{p,\alpha}$, the spaces of all functions such that

$$\|f\|_{p,\alpha} = \left[\int_b^a |f(x)\,e^{-\alpha x}|^p\, dx\right]^{1/p} < \infty \quad ,1 \leq p < \infty$$

…(2.1)

See [7].

we approximated f by a piecewise polynomial of degree at most N.

Definition 1

Let

$S_n(x_1,x_2,\ldots,x_k) = \{ s \in C^{n-1}[a,b] ; s \in \Pi_n(x_{i-1},x_i), i = 1,2,\ldots k+1\}$ where $\Pi_n(x_{i-1},x_i) = (x_0,x_1)(x_1,x_2)\ldots(x_k,x_{k-1})$, $\{x_1, x_2, \ldots, x_k\}$ is a space of spline with simple knots $(x_1, x_2, \ldots, x_k)$, consider $a = x_0 < x_1 < \ldots x_k < x_{k+1} = b$ apartition on interval [a,b] .for k=1,2,…let $S_k = S_n(x_1^k, x_2^k, \ldots, x_k^k)$,for some k knot such that end points [bounded], $a = x_0^k$ and $b = x_{k+1}^k$ for each k. The mesh is denoted by $m_k = \underset{i=0,1,\ldots,k}{Max}\left(x_{i+1}^k - x_i^k\right)$ denote $y_S, S = 1,2,\ldots,n$ the collection of all set

$Y=\{y_s\}$, $0<y_s<\ldots<1$, and $\Delta^{(1)}(Y)$ be the collection of all functions $f, f' \in L_{p,\alpha}[0,1]$ which change monotonicity at the points of y_s.

Set

$$\prod(x)=\prod_{i=1}^{s}(x-yi)$$

…(2.2)

The differentiable function in $L_{p,\alpha}[0,1]$ is in $\Delta^{(1)}(Y)$ If $f'(x)\prod(x)\ge 0,\quad x\in[0,1]$, [4].

Definition 2: [7]

The degree of monotone approximation by polynomials P_n of degree not exceeding n will be denoted by

$$E_n^{(1)}(f,Y)_{p,\alpha}=\inf_{p_n\in\Delta^{(1)}(Y)\cap L_{p,\alpha}}\|f-p_n\|_{p,\alpha}$$

…(2.3)

For $p=\infty$ then $E_n^{(1)}(f,Y)_{\infty,\alpha}=\inf\|f(x)-p_n\|_{\infty,\alpha}$, $x\in[0,1]$

Let

$$\varphi(x)=\sqrt{x(1-x)},\qquad x\in[0,1]$$

…(2.4)

The spaces $L^r_{p,\alpha,\varphi}$,$r\in N$ are the spaces of all functions such that

$$\left[\int_0^1\left|(x)f^{r}(x)e^{-\alpha x}\right|^p dx\right]^{1/p}<\infty$$

…(2.5)

where $f^{(r)}=(f(x)e^{-\alpha x})^{(r)}$ and $\lim_{x\to\frac{1}{2}}\varphi^r(x)f^r(x)=0$.

Definition 3: [3]

For $k\ge 1$ the Ditizian – Totik modules of smoothness is defined by

$$\omega^{\varphi}_{k,r}\left(f^{(r)},t\right)_p=\sup\left\|\varphi^r_{kh}(x)\Delta^k_{h\varphi(x)}f^{(r)}(x)\right\|_p,t\ge 0$$

And $\omega^{\varphi}_{k,r}\left(f^{(r)},t\right)_{p,\alpha}=\sup\left\|\varphi^r_{kh}(x)\Delta^k_{h\varphi(x)}f^{(r)}(x)\right\|_{p,\alpha},t\ge 0$ [7]

…(2.6)

Where k th symmetric difference is defined by

$$\Delta^k_{h\varphi(x)}g(x)=\sum_{j=0}^{k}(-1)^{k+j}\begin{bmatrix}k\\j\end{bmatrix}\quad g\left(x-\frac{(k-j)h\varphi(x)}{2}\right)$$ is

…(2.7)

and the supermum is taken over all $x, x\mp\frac{k}{2}h\varphi(x)\in(0,1)$.

Note that for $f\in L_{p,\alpha}[0,1]$ then $\omega^{\varphi}_{k,0}(f,t)_{p,\alpha}=\omega^{\varphi}_{k}(f,t)_{p,\alpha}$.

Definition 4: [6]

At the points $x_0, x_1, .., x_n$ are defined by $[x_0, x_1, .., x_n; f] = \sum_{j=0}^{n} \frac{f(x_j)}{\prod_{\substack{i=0 \\ i \neq j}} (x_j - x_i)}$

Then $[Z_0, Z_1; g]$ stands for the first divided difference of function g at the knots Z_0 and Z_1, and $[Z_0, Z_1, Z_2; g]$ denotes the second divided difference at the knots Z_0, Z_1 and Z_2.

The Main Result:

In [5] Leviatan and Shevchuk proved that For every $f \in \Delta^{(1)}(Y) \cap C_\varphi^r$ then:

$$E_n^{(1)}(f, Y) \leq \frac{c}{n^r} \omega_{k,r}^{\varphi}\left(f^{(r)}, \frac{1}{n}\right)$$

…(3.1)

where c is a constant independent of n and f .

Also they showed that

Theorem 1: [5]

If $f \in C_\varphi^r \cap \Delta^{(r)}(Y)$ with $r > 2$, then

$$E_n^{(1)}(f, Y) \leq \frac{c(k, r, Y)}{n^r} \quad \omega_{k,r}^{\varphi}\left(f^{(r)}, \frac{1}{n}\right), \quad n \geq k + r$$

…(3.2)

Where C_φ^r , $r \in N$ is the space of functions f, $f^{(r)} \in C^r(0,1)$ for which $\lim_{x \to \frac{1}{2}} \varphi^r(x) f^{(r)}(x) = 0$ and $C_\varphi^0 = C[0,1]$.

Now we prove the following theorem when r ≤ 2 by c(s) denote the different constants which are constants depend only on s, while N(Y) the constants which depend on Y.

Theorem 2: [7]

If $f \in L_{p,\alpha,\varphi}^1 \cap \Delta^{(1)}(Y)$ then

$$E_n^{(1)}(f, Y)_{p,\alpha} \leq \frac{c}{n} \omega_{2,1}^{\varphi}\left(f', \frac{1}{n}\right)_{p,\alpha}, \qquad n \geq N(Y)$$

…(3.3)

An immediate consequence of this theorem is that:

Corollary 1: [7]

If $f \in L_{p,\alpha,\varphi}^2 \cap \Delta^{(1)}(Y)$, then

$$E_n^{(1)}(f, Y)_{p,\alpha} \leq \frac{c}{n^2} \quad \omega_{1,2}^{\varphi}\left(f', \frac{1}{n}\right)_{p,\alpha}, \qquad n \geq N(Y)$$

…(3.4)

Remark:

It should be noted that in the case r=1, k=2 and r=2, k=1 the estimates of form (3.3) and (3.4) in the case $C_r^1 \cap \Delta^{(1)}(Y)$.

Corollary 2: [7]

For each $y_s \in Y$, there is a function $f \in \Delta^{(1)}(Y)$ with $E_n(f)_{p,\alpha} \le \frac{1}{n^2}$, $n \ge 3$

$E_n^{(1)}(f,Y) \le \frac{c}{n^2}$, n ≥ N(Y)

…(3.5)

where $E_n(f)_{p,\alpha}$ is the degree of unconstrained approximation.

Now, let $I=[0,1]$ and put $xj=\frac{1}{n}\left[1-Cos\frac{j\Pi}{n}\right], j=0,1,\dots,n.$

We denote by S_n the set of continuous piecewise polynomials in $L_{p,\alpha}[0,1]$.

We put $I_0=[0,x_0]$ and $I_n=[x_{n-1},x_n]$

Let $L_1(x)=f'(x_1)+(x-x_1)[x_1,2x_1,f']$ be the linear polynomials which interpolates f' at x_1 and $2x_1$. We set

$L_n(x)=f'(x_{n-1})+(x-x_{n-1})\left[x_{n-1}-(1-x_{n-1}),x_{n-1};f'\right]$

Lemma 1: [4, lemma 2]

If $f \in L^1_{p,\varphi} \cap \Delta^{(1)}(Y)$ then there is a continuous piecewise polynomials $\tilde{S}_n \in S_n$ such that

$$\left.\begin{array}{ll} \left\|f-\tilde{S}_n\right\|_p \le \frac{c}{n}\ \omega^{\varphi}_{2,1}\left(f',\frac{1}{n}\right)_p & \\ \Pi(x)\tilde{S}'_n(x)\ge 0 & , x\in\left[x_1,x_{n-1}\right] \\ \tilde{S}'_n(x)=L_n(x) & , \quad x\in I_n \end{array}\right\}$$

…(3.6)

Lemma 2

If $f \in L^1_{p,\alpha,\varphi} \cap \Delta^{(1)}(Y)$ then there is a continuous piecewise polynomials $\tilde{S}_n \in S_n$ such that

$$\left.\begin{array}{l} \left\|f-\tilde{S}_n\right\|_{p,\alpha} \le \frac{c}{n}\ \omega_{2,1}^{\varphi}(f',\frac{1}{n})_{p,\alpha} \\ \Pi(x)\,\tilde{S}_n' \ge 0 \quad , \quad x\in[x_1,x_{n-1}] \\ \tilde{S}_n'(x)=L_n(x) \quad , \quad x\in I_n \end{array}\right\}$$

…(3.7)

Proof of Lemma 2

$$\left\|f-\tilde{S}_n\right\|_{p,\alpha} = \left(\int_0^1 \left|\,\left|f-\tilde{S}_n\right|e^{-\alpha x}\right|^p dx\right)^{1/p} = \left[\int_0^1 \left|fe^{-\alpha x}-\tilde{S}_n e^{-\alpha x}\right|^p dx\right]^{1/p}$$

$$= \left(\int_0^1 \left|g(x)-\tilde{G}_n(x)\right|^p dx\right)^{1/p}$$

$$= \left\|g(x)-\tilde{G}_n(x)\right\|_p$$

such that $g(x)=f(x)e^{-\alpha x}$ and $\tilde{G}_n(x)=\tilde{S}_n e^{-\alpha x}$ where $g(x)\in L^1_{p,\alpha}(Y)$ and $\tilde{G}_n(x)$ is the continuous piecewise polynomial in S_n. Then by lemma 1

$$\left\|f-\tilde{S}_n\right\|_{p,\alpha} = \left\|g-\tilde{G}_n\right\|_p \le \frac{c}{n}\ \omega_{2,1}^{\varphi}\left(g',\frac{1}{n}\right)_p$$

$$= \frac{c}{n}\ \omega_{2,1}^{\varphi}\left(f',\frac{1}{n}\right)_{p,\alpha}$$

Therefore $\left\|f-\tilde{S}_n\right\|_{p,\alpha} \le \frac{c}{n}\ \omega_{2,1}^{\varphi}\left(f',\frac{1}{n}\right)_{p,\alpha}$.

Proof of Theorem (2)

We first take n sufficiently large so that f monotone in I_1 and I_n. Then in view of lemma 2 at most what we have to correct the behavior of $\tilde{S}_n$ on I_1 and I_n while keeping it close to the original function.

A spline polynomial $\tilde{S}_n$ satisfying.

$$\left.\begin{array}{l} S_n(x)=\tilde{S}_n(x) \quad , x\in[x_1,x_{n-1}] \\ \Pi(0)\,\tilde{S}_n(0)\ge 0 \\ \Pi(1)\,\tilde{S}_n(1)\ge 0 \\ \left\|S_n'-\tilde{S}_n'\right\|_{Lp,\alpha(I_1\cup I_n)} \le c_n\ \omega_{2,1}^{\varphi}(f',\frac{1}{n})_{p,\alpha} \end{array}\right\}$$

…(4.1)

Indeed Since $|I_1|,|I_n|\le \frac{c}{n^2}$ then

$$\left\|S'_n-\tilde{S}'_n\right\|_{Lp,\alpha(I_1\cup I_n)} \le \left[\int_{I_1\cup I_n}\left(\int_{I_1\cup I_n}\left|S'_n(t)-\tilde{S}'_n(t)\right|e^{-\alpha x}\,dt\right)^p dx\right]^{1/p}$$

$$=\left[\int_{I_1\cup I_n}\left(\int_{I_1\cup I_n}\left|g-G\right|dt\right)^p dx\right]^{1/p}$$

…(4.2)

Then by holder's inequality we have

$$\left\|S'_n-\tilde{S}'_n\right\|_{Lp,\alpha(I_1\cup I_n)} \le \left[\frac{1}{n^2}\right]^{1/q}\left[\int_{(I_1\cup I_n)}\left\|g-G\right\|^p_{Lp(I_1\cup I_n)}dx\right]^{1/p}$$

$$\le \left[\frac{1}{n^2}\right]^{1/q+1/p}\left\|g-G\right\|^p_{Lp(I_1\cup I_n)}$$

$$=\left[\frac{1}{n^2}\right]^{1/q+1/p}\left\|S'_n-\tilde{S}'_n\right\|^p_{Lp,\alpha(I_1\cup I_n)}$$

$$\le \frac{1}{n^2}\left\|S'_n-\tilde{S}'_n\right\|_{Lp,\alpha(I_1\cup I_n)}$$

From (4.1) and (4.2) we get

$$\left\|S_n-\tilde{S}_n\right\|_{p,\alpha} \le \frac{c}{n^2}\,n\;\omega^{\varphi}_{2,1}\left(f',\frac{1}{n}\right)_{p,\alpha}$$

$$=\frac{c}{n}\;\omega^{\varphi}_{2,1}\left(f',\frac{1}{n}\right)_{p,\alpha}$$

Which combined with (3.6) and (4.3) implies.

$$E_n^{(1)}(f,Y)_{p,\alpha}=\left\|f-S_n\right\|_{p,\alpha}\le\left\|f-\tilde{S}_n\right\|_{p,\alpha}+\left\|S_n-\tilde{S}_n\right\|_{p,\alpha}$$

$$\le\left\|f-\tilde{S}_n\right\|_{p,\alpha}+\left\|S_n-\tilde{S}_n\right\|_{Lp,\alpha(L_1\cup L_2)}+\left\|S_n-\tilde{S}_n\right\|_{Lp,\alpha[x_1,x_{n-1}]}$$

$$\le\frac{c}{n}\;\omega^{\varphi}_{2,1}\left(f',\frac{1}{n}\right)_{p,\alpha}+0+\frac{c}{n}\;\omega^{\varphi}_{2,1}\left(f',\frac{1}{n}\right)_{p,\alpha}$$

$$=\frac{c}{n}\;\omega^{\varphi}_{2,1}\left(f',\frac{1}{n}\right)_{p,\alpha}$$

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الخلاصة

الغرض من هذا البحث هو دراسة درجة أفضل تقريب للدوال الغير مقيدة في فضاء الوزن $L_{p,\alpha}$, $(1 \le p \le \infty)$, $\alpha > 0$.

الكلمات المفتاحية : فضاء الوزن، الدوال غير المقيدة، التقريب الرتيب.