Ibn Al-Haitham Journal for Pure and Applied Science | مجلة إبن الهيثم للعلوم الصرفة و التطبيقية
No. 3 Vol. 25 Year 2012 | العدد 3 المجلد 25 السنة 2012

The Reduction of Resolution of Weyl Module from Characteristic-Free Resolution in Case (4,4,3)

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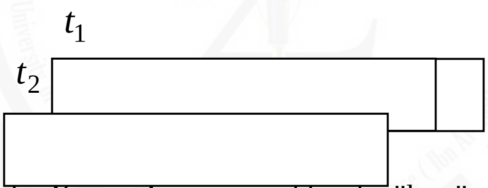
Received in: 8 January 2012 Accepted in: 22 April 2012

Abstract

In this paper we study the relation between the resolution of Weyl Module $K_{(4,4,3)}F$ in characteristic-free mode and in the Lascoux mode (characteristic zero), more precisely we obtain the Lascoux resolution of $K_{(4,4,3)}F$ in characteristic zero as an application of the resolution of $K_{(4,4,3)}F$ in characteristic-free.
Key word : Resolution**,** Weyl module**,** Lascoux mode, divided power, characteristic-free.

Introduction

Let R be a commutative ring with 1 and F be free R-module by D_nF we mean the divided power of degree n. The resolution Res[p, q, r, t_1, t_2] of Weyl module $K_{\lambda/\mu}F$ associated to the three-rowed skew-shape $(p+t_1+t_2, q+t_2, r)/(t_1+t_2, t_2, 0)$, namely , the shape represented by the diagram



In general, the Weyl module $K_{\lambda/\mu}$ is presented by the "box" map

$$\sum_{k>0} D_{p+t_1+k}F \otimes D_{q-t_1-k} \otimes D_rF$$

$$\longrightarrow D_pF \otimes D_qF \otimes D_rF \xrightarrow{d'_{\lambda/\mu}} K_{\lambda/\mu}$$

$$\sum_{\ell>0} D_pF \otimes D_{q+t_2+\ell} \otimes D_{r-t_2-\ell}F$$

where the maps $\sum_{k>0} D_{p+t_1+k}F \otimes D_{q-t_1-k} \otimes D_rF \longrightarrow D_pF \otimes D_qF \otimes D_rF$ may be interpreted as K[th] divided power of the place polarization from place 1 to place2(i.e. $\partial_{32}^{(k)}$),the maps $\sum D_pF \otimes D_{q+t_2+\ell} \otimes D_{r-t_2-\ell}F \longrightarrow D_pF \otimes D_qF \otimes D_rF$ may be place 2 interpreted as ℓ^{th} divided power of the place polarization from place 2 to 3 (i.e. $\partial_{32}^{(\ell)}$) [1].

Buchsbaum in [1], Hassan in [2] studied the resolution of Weyl module in the case of the partition (2,2,2) and (3,3,3) respectively. In section two of this paper we review the terms of characteristic-free resolution of Weyl module in the case of the partition (4,4,3).

In section three we apply this resolution to the Lascoux resolution in the same case by using the way in [1] and [2] with capelli identities [3].

Note: We have to mention that we shall use D_n instead of D_nF to refer to divided power algebra of degree n.

Characteristic-Free Resolution of the Partition (4,4,3)

In this section, we find the terms of the resolution of Weyl module in the case of the partition (4,4,3). In general the terms of the resolution of Weyl module in the case of a three-rowed partition (p,q,r) which appeared in [3] are

$$\mathrm{Res}([p,q;o])\otimes D_r \oplus \sum_{\ell\geq 0} Z_{32}^{(\ell+1)} y\,\mathrm{Res}([p,q+\ell+1;\ell+1])\otimes D_{r-\ell-1} \oplus$$

$$\sum_{\ell_1\geq 0,\ell_2\geq \ell_1} \underline{Z}_{32}^{(\ell_2+1)}\underline{Z}_{31}^{(\ell_1+1)} z\,\mathrm{Res}([p+\ell_1+1,q+\ell_2+1;\ell_2-\ell_1])\otimes D_{r-(\ell_1+\ell_2+2)}$$

where x, y and z stand for the separator variables, and the boundary map is $\partial_x+\partial_y+\partial_z$. Let again Bar(M,A,S) be the free bar module on the set $S=\{x,y,z\}$ consisting of three separators *x*, *y* and *z* where A is the free associative (non-commutative) algebra generated by Z_{21}, Z_{32} and Z_{31} and their divided powers with the following relations:

$Z_{32}^{(a)}Z_{31}^{(b)}=Z_{31}^{(b)}Z_{32}^{(a)}$ and $Z_{21}^{(a)}Z_{31}^{(b)}=Z_{31}^{(b)}Z_{21}^{(a)}$ and the module M is the direct sum of tensor product of divided power module $D_q\otimes D_r$ for suitable *p*, *q* and *r* with the action of Z_{21}, Z_{32} and Z_{31} and their divided powers.

We'll consider the case when $p=4,\ q=4$ and $r=3$. In this case, we see that the index ℓ in the first sum runs from 0 to 2, while the indices ℓ_1 and ℓ_2 in the second sum are very restricted : ℓ_1 must be 0, and ℓ_2 can be only 0 and 1 (since $\ell_2\geq\ell_1$ and $\ell_1+\ell_2\leq 1$). These comments are true whatever the values of *p* and *q*, then in this case we have

$$\mathrm{Res}([4,4;0])\otimes D_3 \oplus \sum_{\ell\geq 0} Z_{32}^{(\ell+1)} y\,\mathrm{Res}([4,4+\ell+1;\ell+1])\otimes D_{3-\ell-1} \oplus$$

$$\sum_{\ell_1\geq 0,\ell_2\geq \ell_1} \underline{Z}_{32}^{(\ell_2+1)} y\,\mathrm{Res}([4+\ell_1+1,4+\ell_2+1;\ell_2-\ell_1])\otimes D_{3-(\ell_1+\ell_2+2)}$$

So

$$\sum_{\ell\geq 0} \underline{Z}_{32}^{(\ell+1)} y\,\mathrm{Res}([4,4+\ell+1;\ell+1])\otimes D_{3-\ell-1} =$$

$$\underline{Z}_{32} y\,\mathrm{Res}([4,5;1])\otimes D_2 \oplus \underline{Z}_{32}^{(2)} y\,\mathrm{Res}([4,6;2])\otimes D_1 \oplus \underline{Z}_{32}^{(3)}\,\mathrm{Res}([4,7;3])\otimes D_0$$

Where

$\underline{Z}_{32}^{(2)} y$ is the complex $0\rightarrow Z_{32}yZ_{32}\rightarrow Z_{32}^{(2)}y\rightarrow 0$

and

$\underline{Z}_{32}^{(3)} y$ is the complex

$$0\rightarrow Z_{32}yZ_{32}yZ_{32}y\rightarrow Z_{32}^{(2)}yZ_{32}y\oplus Z_{32}yZ_{32}^{(2)}y\rightarrow Z_{32}^{(3)}y\rightarrow 0$$

Then in this case we have the following terms

- In dimension zero (M_0) we have $D_4\otimes D_4\otimes D_3$

- In dimension one (M_1) we have

$Z_{21}^{(b)} x D_{4+b} \otimes D_{4-b} \otimes D_3$ with b=1,2,3,4 and $Z_{32}^{(b)} y D_4 \otimes D_{4+b} \otimes D_{3-b}$ with b =1,2,3

- In dimension two (M2) we have the sum of the following terms

- $Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x D_{4+|b|} \otimes D_{4-|b|} \otimes D_3$; with $|b| = b_1 + b_2 = 2,3,4.$
- $Z_{32} y Z_{21}^{(b)} x D_{4+b} \otimes D_{5-b} \otimes D_2$; with b= 2,3,4,5.
- $Z_{32}^{(b_1)} y Z_{32}^{(b_2)} x D_4 \otimes D_{4+|b|} \otimes D_{3-|b|}$; with $|b| = b_1 + b_2 = 2,3.$
- $Z_{32}^{(2)} y Z_{21}^{(b)} x D_{4+b} \otimes D_{6-b} \otimes D_1$; with b= 3,4,5,6.
- $Z_{32}^{(3)} y Z_{21}^{(b)} x D_{4+b} \otimes D_{7-b} \otimes D_0$; with b= 4,5,6,7.
- $Z_{32}^{(b)} y Z_{31} z D_5 \otimes D_{4+b} \otimes D_{2-b}$; with b= 1,2.

- In dimension three (M_3) we have the sum of the following terms:

- $Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x D_{4+|b|} \otimes D_{4-|b|} \otimes D_3$; with $|b| = b_1 + b_2 + b_3 = 3,4.$
- $Z_{32} y Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x D_{4+|b|} \otimes D_{5-|b|} \otimes D_2$; with $|b| = b_1 + b_2 = 3,4,5$ and $b_1 \geq 2$
- $Z_{32}^{(2)} y Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x D_{4+|b|} \otimes D_{6-|b|} \otimes D_1$; with $|b| = b_1 + b_2 = 4,5,6$ and $b_1 \geq 3$
- $Z_{32} y Z_{32} y Z_{21}^{(b)} x D_{4+b} \otimes D_{6-b} \otimes D_1$; with b= 3,4,5,6.
- $Z_{32}^{(3)} y Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x D_{4+|b|} \otimes D_{7-|b|} \otimes D_0$; with $|b| = b_1 + b_2 = 5,6,7$ and $b_1 \geq 4$
- $Z_{32}^{(b_1)} y Z_{32}^{(b_2)} x Z_{21}^{(k)} x D_{4+k} \otimes D_{7-k} \otimes D_0$; with $b_1 + b_2 = 3$ and $k = 4,5,6,7$
- $Z_{32} y Z_{32} y Z_{32} y D_4 \otimes D_7 \otimes D_0$
- $Z_{32} y Z_{31} z Z_{21}^{(b)} x D_{5+b} \otimes D_{5-b} \otimes D_1$; with b= 1,2,3,4,5.
- $Z_{32}^{(2)} y Z_{31} z Z_{21}^{(b)} x D_{5+b} \otimes D_{6-b} \otimes D_0$; with b= 2,3,4,5,6.
- $Z_{32} y Z_{32} y Z_{31} z D_5 \otimes D_6 \otimes D_0$.

- In dimension four (M_4) we have the sum of the following terms:

- $Z_{21} x Z_{21} x Z_{21} x Z_{21} x D_8 \otimes D_0 \otimes D_3$
- $Z_{32} y Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x D_{4+|b|} \otimes D_{5-|b|} \otimes D_2$; with $|b| = b_1 + b_2 + b_3 = 4,5$ and $b_1 \geq 2$
- $Z_{32}^{(2)} y Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x Z_{21}^{(b_3)} x D_{4+|b|} \otimes D_{6-|b|} \otimes D_1$; with $|b| = b_1 + b_2 + b_3 = 5,6$ and $b_1 \geq 3$.

- $Z_{32}yZ_{32}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xD_{4+|b|}\otimes D_{6-|b|}\otimes D_1$; with $|b|=b_1+b_2+b_3=4,5,6$ and $b_1\geq 3$.
- $Z_{32}^{(3)}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xD_{4+|b|}\otimes D_{7-|b|}\otimes D_0$; with $|b|=b_1+b_2+b_3=6,7$ and $b_1\geq 4$.
- $Z_{32}^{(c_1)}yZ_{32}^{(c_2)}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xD_{4+|b|}\otimes D_{7-|b|}\otimes D_0$; with $c_1+c_2=3$, $|b|=b_1+b_2+b_3=5,6,7$ and $b_1\geq 4$.
- $Z_{32}yZ_{32}yZ_{32}yZ_{21}^{(b)}xD_{4+b}\otimes D_{7-b}\otimes D_0$; with b= 4,5,6,7.
- $Z_{32}yZ_{31}zZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xD_{5+|b|}\otimes D_{5-|b|}\otimes D_1$; with $|b|=b_1+b_2=2,3,4,5$.
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xD_{5+|b|}\otimes D_{6-|b|}\otimes D_0$; with $|b|=b_1+b_2=3,4,5,6$ and $b_1\geq 2$.
- $Z_{32}yZ_{32}yZ_{31}zZ_{21}^{(b)}xD_{5+b}\otimes D_{6-b}\otimes D_0$; with b= 2,3,4,5,6.

○ In dimension five (M_5) we have the sum of the following terms:

- $Z_{32}^{(2)}yZ_{21}^{(3)}xZ_{21}xZ_{21}xZ_{21}xD_{10}\otimes D_0\otimes D_1$
- $Z_{32}yZ_{32}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xD_{4+|b|}\otimes D_{6-|b|}\otimes D_1$; with $|b|=b_1+b_2+b_3=5,6$ and $b_1\geq 3$.
- $Z_{32}^{(3)}yZ_{21}^{(4)}xZ_{21}xZ_{21}xZ_{21}xD_{11}\otimes D_0\otimes D_0$
- $Z_{32}^{(c_1)}yZ_{32}^{(c_2)}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xD_{4+|b|}\otimes D_{7-|b|}\otimes D_0$; with $c_1+c_2=3$, $|b|=b_1+b_2+b_3=6,7$ and $b_1\geq 4$.
- $Z_{32}yZ_{32}yZ_{32}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xD_{4+|b|}\otimes D_{7-|b|}\otimes D_0$; with $|b|=b_1+b_2+b_3=5,6,7$ and $b_1\geq 4$.
- $Z_{32}yZ_{31}zZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xD_{5+|b|}\otimes D_{5-|b|}\otimes D_1$; with $|b|=b_1+b_2+b_3=3,4,5$.
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xD_{5+|b|}\otimes D_{6-|b|}\otimes D_0$; with $|b|=b_1+b_2+b_3=4,5,6$ and $b_1\geq 2$.
- $Z_{32}yZ_{32}yZ_{31}zZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xD_{5+|b|}\otimes D_{6-|b|}\otimes D_0$; with $|b|=b_1+b_2=3,4,5,6$ and $b_1\geq 2$.

○ In dimension six (M_6) we have the sum of the following terms:

- $Z_{32}^{(c_1)}yZ_{32}^{(c_2)}yZ_{21}^{(4)}xZ_{21}xZ_{21}xZ_{21}xD_{11}\otimes D_0\otimes D_0$; with $c_1+c_2=3$.

- $Z_{32}yZ_{32}yZ_{32}yZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xD_{4+|b|}\otimes D_{7-|b|}\otimes D_0;$

with $|b|=b_1+b_2+b_3=6,7$ and $b_1\geq 4$.

- $Z_{32}yZ_{31}zZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xZ_{21}^{(b_4)}xD_{5+|b|}\otimes D_{5-|b|}\otimes D_1$; with

$|b|=\sum_{i=1}^{4}b_i=4,5$.

- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xZ_{21}^{(b_4)}xD_{5+|b|}\otimes D_{6-|b|}\otimes D_0$; with

$|b|=\sum_{i=1}^{4}b_i=5,6$ and $b_1\geq 2$.

- $Z_{32}yZ_{32}yZ_{31}zZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xD_{5+|b|}\otimes D_{6-|b|}\otimes D_0;$

with $|b|=b_1+b_2+b_3=4,5,6$ and $b_1\geq 2$.

○ In dimension seven (M_7) we have the sum of the following terms:

- $Z_{32}yZ_{31}zZ_{21}xZ_{21}xZ_{21}xZ_{21}xZ_{21}xD_{10}\otimes D_0\otimes D_1$.
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(2)}xZ_{21}xZ_{21}xZ_{21}xZ_{21}xD_{11}\otimes D_0\otimes D_0$.
- $Z_{32}yZ_{32}yZ_{31}zZ_{21}^{(b_1)}xZ_{21}^{(b_2)}xZ_{21}^{(b_3)}xZ_{21}^{(b_4)}xD_{5+|b|}\otimes D_{6-|b|}\otimes D_0;$ with

$|b|=\sum_{i=1}^{4}b_i=5,6$ and $b_1\geq 2$.

And finally in dimension eight (M_8)we have

- $Z_{32}yZ_{32}yZ_{31}zZ_{21}^{(2)}xZ_{21}xZ_{21}xZ_{21}xZ_{21}xD_{11}\otimes D_0\otimes D_0$.

As in [3], it is necessary to introduce a quotient of Bar complex modulo the Capelli identities relations; the proof these relation are compatible with the boundary map $\partial_x+\partial_y+\partial_z$ is complicated and proved in [3].

Lascoux resolution for (4,4,3)

The Lascoux resolution of the Weyl module associated to the partition (4,4,3) looks like this

$$0\to D_6\otimes D_4\oplus D_1\to \begin{matrix} D_5\otimes D_5\oplus D_1\\ \oplus \\ D_6\otimes D_3\oplus D_2\end{matrix} \to \begin{matrix} D_4\otimes D_5\oplus D_2\\ \oplus \\ D_5\otimes D_3\oplus D_3\end{matrix}\to D_4\otimes D_4\oplus D_3\to 0$$

where the position of the terms of the complex determined by the length of the permutations to which they correspond. The correspondence between the terms of the resolution above and permutations is as follows

$D_4\otimes D_4\oplus D_3\Leftrightarrow$ identity

$$D_5\otimes D_2\oplus D_4\Leftrightarrow (1\ 2)$$
$$D_3\otimes D_5\oplus D_3\Leftrightarrow (2\ 3)$$

$D_5\otimes D_5\oplus D_1\Leftrightarrow (1\ 2\ 3)$

$D_6 \otimes D_2 \oplus D_3 \Leftrightarrow (2\ 1\ 3)$

$$D_6 \otimes D_4 \oplus D_1 \Leftrightarrow (1\ 3)$$

Now, the terms can be presented as below, following Buchsbaum method [1].

$M_0 = A_0$

$M_1 = A_1 \oplus B_1$

$M_2 = A_2 \oplus B_2$

$M_3 = A_3 \oplus B_3$

and

$M_j = B_j$; for j=4,5,6,7,8.

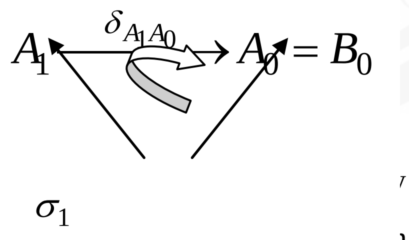
Where the A_s are the sums of the Lascoux terms, and the B_s are the sums of the others.

Now, we define the map σ_1 from B_1 to A_1 as follows

- $Z_{21}^{(2)} x(v) \mapsto \frac{1}{2} Z_{21} x \partial_{21}(v)$; where $v \in D_6 \otimes D_2 \oplus D_3$
- $Z_{21}^{(3)} x(v) \mapsto \frac{1}{3} Z_{21} x \partial_{21}^{(2)}(v)$; where $v \in D_7 \otimes D_1 \oplus D_3$
- $Z_{21}^{(4)} x(v) \mapsto \frac{1}{4} Z_{21} x \partial_{21}^{(3)}(v)$; where $v \in D_8 \otimes D_0 \oplus D_3$
- $Z_{32}^{(2)} y(v) \mapsto \frac{1}{2} Z_{32} y \partial_{32}(v)$; where $v \in D_4 \otimes D_6 \oplus D_1$
- $Z_{32}^{(3)} y(v) \mapsto \frac{1}{3} Z_{32} y \partial_{32}^{(2)}(v)$; where $v \in D_4 \otimes D_7 \oplus D_0$

We should point out that the map σ_1 satisfies the identity:

$\delta_{A_1 A_0} \circ \sigma_1 = \delta_{B_1 B_0}$ …(3.1)



where by $\delta_{A_1 A_0}$ we mear … of the fat complex which conveys A_1 to A_0. We will use no … we can define $\partial_1 : A_1 \to A_0$ as $\partial_1 = \delta_{A_1 A_0}$.

It is easy to show that ∂_1 which we defined above satisfies the condition (3.1), for example:

$$(\delta_{A_1 A_0} \circ \sigma_1)(Z_{21}^{(3)} x(v)) = \delta_{A_1 A_0}\left(\frac{1}{3} Z_{21} x \partial_{21}^{(2)}(v)\right) = \frac{1}{3}(\partial_{21} \partial_{21}^{(2)}(v)) = \partial_{21}^{(3)}(v)$$

$$= \delta_{B_1 B_0}(Z_{21}^{(3)} x(v)).$$

At this point we are in position to define

$\partial_2 : A_2 \to A_1$

By $\partial_2 = \delta_{A_2 A_1} + \sigma_1 \delta_{A_2 B_1}$

Proposition (3.1): The composition $\partial_1 \circ \partial_2 = 0$ [1],[2] .□

Now, we have to define a map
$\sigma_2 : B_2 \to A_2$
Such that

$$\delta_{B_2A_1} + \sigma_1 \circ \delta_{B_2B_1} = (\delta_{A_2A_1} + \sigma_1 \circ \delta_{A_2B_1}) \circ \sigma_2 \quad \text{.........(3.2)}$$

We define this map as follows:

- $Z_{21}xZ_{21}xv \mapsto 0$; where $v \in D_6 \otimes D_2 \oplus D_3$
- $Z_{21}^{(2)}xZ_{21}xv \mapsto 0$; where $v \in D_7 \otimes D_1 \oplus D_3$
- $Z_{21}xZ_{21}^{(2)}xv \mapsto 0$; where $v \in D_7 \otimes D_1 \oplus D_3$
- $Z_{21}^{(3)}xZ_{21}xv \mapsto 0$; where $v \in D_8 \otimes D_0 \oplus D_3$
- $Z_{21}^{(2)}xZ_{21}^{(2)}xv \mapsto 0$; where $v \in D_8 \otimes D_0 \oplus D_3$
- $Z_{21}xZ_{21}^{(3)}xv \mapsto 0$; where $v \in D_8 \otimes D_0 \oplus D_3$
- $Z_{32}yZ_{21}^{(3)}xv \mapsto \frac{1}{3}Z_{32}yZ_{21}^{(3)}x\partial_{21}(v)$; where $v \in D_7 \otimes D_2 \oplus D_2$
- $Z_{32}yZ_{21}^{(4)}xv \mapsto \frac{1}{6}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}(v)$; where $v \in D_8 \otimes D_1 \oplus D_2$
- $Z_{32}yZ_{21}^{(5)}xv \mapsto \frac{1}{10}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}(v)$; where $v \in D_9 \otimes D_0 \oplus D_2$
- $Z_{32}yZ_{32}yv \mapsto 0$; where $v \in D_4 \otimes D_6 \oplus D_1$
- $Z_{32}^{(2)}yZ_{21}^{(3)}xv \mapsto \frac{1}{3}Z_{32}yZ_{21}^{(2)}x\partial_{31}(v) - \frac{1}{3}Z_{32}yZ_{31}z\partial_{21}^{(2)}(v)$;where $v \in D_7 \otimes D_3 \oplus D_1$
- $Z_{32}^{(2)}yZ_{21}^{(4)}xv \mapsto \frac{1}{6}Z_{32}yZ_{21}^{(2)}x\partial_{21}\partial_{31}(v) - \frac{1}{12}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}(v)$;where $v \in D_8 \otimes D_2 \oplus D_1$
- $Z_{32}^{(2)}yZ_{21}^{(5)}xv \mapsto \frac{1}{30}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}(v) - \frac{1}{5}Z_{32}yZ_{31}z\partial_{21}^{(4)}(v)$;where $v \in D_9 \otimes D_1 \oplus D_1$
- $Z_{32}^{(2)}yZ_{21}^{(6)}xv \mapsto \frac{1}{20}Z_{32}yZ_{21}^{(2)}x\partial_{32}\partial_{21}^{(4)}(v)$; where $v \in D_{10} \otimes D_0 \oplus D_1$
- $Z_{32}^{(3)}yZ_{21}^{(4)}xv \mapsto \frac{1}{6}Z_{32}yZ_{31}^{(2)}\partial_{31}^{(2)}(v) + \frac{1}{12}Z_{32}yZ_{21}^{(2)}x\partial_{32}\partial_{21}\partial_{31}(v)$ $-\frac{1}{12}Z_{32}yZ_{31}z\partial_{21}^{(3)}\partial_{32}(v)$; where $v \in D_8 \otimes D_3 \oplus D_0$

- $Z_{32}^{(3)}yZ_{21}^{(5)}xv \mapsto \frac{1}{9}Z_{32}yZ_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v)+\frac{1}{18}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)$

$+\frac{1}{30}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)$; where $v \in D_9 \otimes D_2 \oplus D_0$

- $Z_{32}^{(3)}yZ_{21}^{(6)}xv \mapsto -\frac{1}{9}Z_{32}yZ_{31}z\partial_{21}^{(4)}\partial_{31}(v)-\frac{1}{36}Z_{32}yZ_{31}z\partial_{21}^{(5)}\partial_{32}(v)$ where $v \in D_{10} \otimes D_1 \oplus D_0$
- $Z_{32}^{(3)}yZ_{21}^{(7)}xv \mapsto \frac{1}{30}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v)$; where $v \in D_{11} \otimes D_0 \oplus D_0$
- $Z_{32}^{(2)}yZ_{32}yv \mapsto 0$; where $v \in D_4 \otimes D_7 \oplus D_0$
- $Z_{32}yZ_{32}^{(2)}yv \mapsto 0$; where $v \in D_4 \otimes D_7 \oplus D_0$
- $Z_{32}^{(3)}yZ_{31}zv \mapsto \frac{1}{3}Z_{32}yZ_{31}z\partial_{32}(v)$; where $v \in D_5 \otimes D_6 \oplus D_0$

It is easy to show that σ_2 which is defined above satisfies the condition (3.2), for example we chose one of them

$(\delta_{B_2A_1}+\sigma_1 \circ \delta_{B_2B_1})(Z_{32}^{(3)}yZ_{21}^{(4)}xv)$

$=\sigma_1(Z_{21}^{(4)}x\partial_{32}^{(3)}v)+\sigma_1(Z_{21}^{(3)}x\partial_{32}^{(2)}\partial_{31}(v))+\sigma_1(Z_{21}^{(2)}x\partial_{32}\partial_{31}^{(2)}(v))$

$+Z_{21}x\partial_{31}^{(31)}(v)-\sigma_1(Z_{32}^{(3)}x\partial_{21}^{(4)}(v))$

$=\frac{1}{4}Z_{21}x\partial_{21}^{(3)}\partial_{32}^{(3)}(v)+\frac{1}{3}Z_{21}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v)+\frac{1}{2}Z_{21}x\partial_{21}\partial_{32}\partial_{31}^{(2)}(v)$

$+Z_{21}x\partial_{31}^{(3)}(v)-\frac{1}{3}Z_{32}y\partial_{32}^{(2)}\partial_{21}^{(4)}(v)$

$=\frac{1}{4}Z_{21}x\partial_{21}^{(3)}\partial_{32}^{(3)}(v)+\frac{1}{3}Z_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}\partial_{31}(v)+\frac{1}{2}Z_{21}x\partial_{21}\partial_{32}\partial_{31}^{(2)}(v)$

$+Z_{21}x\partial_{31}^{(3)}(v)-\frac{1}{3}Z_{32}y\partial_{21}^{(4)}\partial_{32}^{(2)}(v)$

$-\frac{1}{3}Z_{32}y\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-\frac{1}{3}Z_{32}y\partial_{21}^{(3)}\partial_{32}\partial_{31}(v)-\frac{1}{3}Z_{32}y\partial_{21}^{(2)}\partial_{31}^{(2)}(v)$ and

$(\delta_{A_2A_1}+\sigma_1 \circ \delta_{A_2B_1})(\frac{1}{6}Z_{32}yZ_{21}^{(2)}x\partial_{31}^{(2)}(v)+\frac{1}{12}Z_{32}yZ_{21}^{(2)}x\partial_{31}\partial_{21}(v)$

$-\frac{1}{12}Z_{32}yZ_{31}z\partial_{21}^{(3)}\partial_{31}(v))$

$=\sigma_1(\frac{1}{6}Z_{21}^{(2)}x\partial_{32}\partial_{31}^{(2)}(v))+\frac{1}{2}Z_{21}x\partial_{31}^{(3)}(v)-\frac{1}{6}Z_{32}y\partial_{21}^{(2)}\partial_{31}^{(2)}(v)$

$+\sigma_1(\frac{1}{6}Z_{21}^{(2)}x\partial_{32}^{(2)}\partial_{21}\partial_{31}(v))+\frac{1}{6}Z_{21}x\partial_{32}\partial_{21}\partial_{31}^{(2)}(v)-\frac{1}{12}Z_{21}x\partial_{21}^{(2)}\partial_{32}\partial_{21}\partial_{31}(v)$

$$-\sigma_1(\frac{1}{3}Z_{32}^{(2)}y\partial_{21}^{(4)}\partial_{32}(v))+\frac{1}{12}Z_{21}x\partial_{32}^{(2)}\partial_{21}^{(3)}\partial_{32}(v)+\frac{1}{12}Z_{32}y\partial_{21}^{(3)}\partial_{32}\partial_{31}(v)$$

$$=\frac{1}{12}Z_{21}x\partial_{21}\partial_{32}\partial_{31}^{(2)}(v)+\frac{1}{12}Z_{21}x\partial_{31}^{(3)}(v)-\frac{1}{6}Z_{32}y\partial_{21}^{(2)}\partial_{31}^{(2)}(v)$$

$$+\frac{1}{6}Z_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}\partial_{31}(v)+\frac{1}{6}Z_{21}x\partial_{21}\partial_{32}\partial_{31}^{(2)}(v)+\frac{1}{6}Z_{21}x\partial_{21}\partial_{32}\partial_{31}^{(2)}(v)$$

$$+\frac{1}{2}Z_{21}x\partial_{31}^{(3)}(v)-\frac{1}{4}Z_{32}y\partial_{21}^{(3)}\partial_{32}\partial_{31}(v)-\frac{1}{6}Z_{32}y\partial_{21}^{(2)}\partial_{31}^{(2)}(v)$$

$$-\frac{1}{3}Z_{32}y\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-\frac{1}{6}Z_{32}y\partial_{21}^{(3)}\partial_{32}\partial_{31}(v)+\frac{1}{4}Z_{21}x\partial_{21}^{(3)}\partial_{32}^{(3)}(v)$$

$$+\frac{1}{6}Z_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}\partial_{31}(v)+\frac{1}{12}Z_{21}x\partial_{21}\partial_{32}\partial_{31}^{(2)}(v)+\frac{1}{12}Z_{32}y\partial_{21}^{(3)}\partial_{32}\partial_{31}(v)$$

$$=\frac{1}{2}Z_{21}x\partial_{21}\partial_{32}\partial_{31}^{(2)}(v)+Z_{21}x\partial_{31}^{(3)}(v)-\frac{1}{3}Z_{32}y\partial_{21}^{(2)}\partial_{31}^{(2)}(v)$$

$$+\frac{1}{3}Z_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}\partial_{31}(v)-\frac{1}{3}Z_{32}y\partial_{21}^{(3)}\partial_{32}\partial_{31}(v)$$

$$-\frac{1}{3}Z_{32}y\partial_{21}^{(4)}\partial_{32}^{(2)}(v)+\frac{1}{4}Z_{21}x\partial_{21}^{(3)}\partial_{32}^{(3)}(v)\ \square$$

Proposition (3.2) [1],[2] **:** we have exactness at A_i .□

Now by using σ_2 we can also define
$\partial_3 : A_3 \to A_2$ by $\partial_3 = \delta_{A_3A_2} + \sigma_2 \circ \delta_{A_3B_2}$

Proposition (3.3) [1],[2] **:** $\partial_2 \circ \partial_3 = 0$.□

We need to define $\sigma_3 : B_3 \to A_3$ which satisfying

$$\delta_{B_3A_2} + \sigma_2 \circ \delta_{B_3B_2} = (\delta_{A_3A_2} + \sigma_2 \circ \delta_{A_3B_2}) \circ \sigma_2 \ldots (3.3)$$

As follows

- $Z_{21}xZ_{21}xZ_{21}xv \mapsto 0;$ where $v \in D_7 \otimes D_1 \oplus D_3$
- $Z_{21}^{(2)}xZ_{21}xZ_{21}xv \mapsto 0;$ where $v \in D_8 \otimes D_0 \oplus D_3$
- $Z_{21}xZ_{21}^{(2)}xZ_{21}xv \mapsto 0;$ where $v \in D_8 \otimes D_0 \oplus D_3$
- $Z_{21}xZ_{21}xZ_{21}^{(3)}xv \mapsto 0;$ where $v \in D_8 \otimes D_0 \oplus D_3$
- $Z_{32}yZ_{21}^{(2)}xZ_{21}xv \mapsto 0;$ where $v \in D_7 \otimes D_2 \oplus D_2$
- $Z_{32}yZ_{21}^{(3)}xZ_{21}xv \mapsto 0;$ where $v \in D_8 \otimes D_1 \oplus D_2$
- $Z_{32}yZ_{21}^{(2)}xZ_{21}^{(2)}xv \mapsto 0;$ where $v \in D_8 \otimes D_1 \oplus D_2$

- $Z_{32}yZ_{21}^{(4)}xZ_{21}xv \mapsto 0$; where $v \in D_9 \otimes D_0 \oplus D_2$
- $Z_{32}yZ_{21}^{(3)}xZ_{21}^{(2)}xv \mapsto 0$; where $v \in D_9 \otimes D_0 \oplus D_2$
- $Z_{32}yZ_{21}^{(2)}xZ_{21}^{(2)}xv \mapsto 0$; where $v \in D_9 \otimes D_0 \oplus D_2$
- $Z_{32}^{(2)}yZ_{21}^{(3)}xZ_{21}xv \mapsto -\frac{1}{3}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(2)}(v)$; where $v \in D_8 \otimes D_2 \oplus D_1$
- $Z_{32}^{(2)}yZ_{21}^{(4)}xv \mapsto \frac{1}{4}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(3)}(v)$; where $v \in D_9 \otimes D_1 \oplus D_1$
- $Z_{32}^{(2)}yZ_{21}^{(3)}xZ_{21}^{(2)}xv \mapsto 0$ where $v \in D_9 \otimes D_1 \oplus D_1$
- $Z_{32}^{(2)}yZ_{21}^{(5)}xZ_{21}xv \mapsto -\frac{1}{5}Z_{32}^{(2)}yZ_{31}^{(5)}zZ_{21}x\partial_{21}^{(4)}(v)$ where $v \in D_{10} \otimes D_0 \oplus D_1$
- $Z_{32}^{(2)}yZ_{21}^{(4)}xZ_{21}^{(2)}xv \mapsto -\frac{1}{4}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(4)}(v)$ where $v \in D_{10} \otimes D_0 \oplus D_1$
- $Z_{32}^{(2)}yZ_{21}^{(3)}xZ_{21}^{(3)}xv \mapsto -\frac{2}{3}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(4)}(v)$ where $v \in D_{10} \otimes D_0 \oplus D_1$
- $Z_{32}yZ_{32}yZ_{21}^{(3)}xv \mapsto -\frac{1}{3}Z_{32}yZ_{31}zZ_{21}x\partial_{21}(v)$ where $v \in D_7 \otimes D_3 \oplus D_1$
- $Z_{32}yZ_{32}yZ_{21}^{(4)}xv \mapsto 0$ where $v \in D_8 \otimes D_1 \oplus D_2$
- $Z_{32}yZ_{32}yZ_{21}^{(5)}xv \mapsto \frac{1}{10}Z_{32}yZ_{31}zZ_{21}^{(2)}x\partial_{21}^{(3)}(v)$; with $v \in D_9 \otimes D_1 \oplus D_1$
- $Z_{32}yZ_{32}yZ_{21}^{(6)}xv \mapsto 0$ with $v \in D_{10} \otimes D_0 \oplus D_1$
- $Z_{32}^{(2)}yZ_{21}^{(4)}xZ_{21}xv \mapsto Z_{32}^{(2)}yZ_{21}^{(5)}xZ_{21}xv \mapsto \frac{1}{6}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(3)}\partial_{31}(v) +$

$\frac{1}{30}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(4)}\partial_{32}(v)$; with $v \in D_{10} \otimes D_1 \oplus D_0$

- $Z_{32}^{(2)}yZ_{21}^{(4)}xZ_{21}^{(2)}xv \mapsto 0$; with $v \in D_{10} \otimes D_0 \oplus D_1$
- $Z_{32}^{(2)}yZ_{21}^{(3)}xZ_{21}^{(3)}xv \mapsto \frac{2}{3}Z_{32}yZ_{31}zZ_{21}x(v)$ with $v \in D_{10} \otimes D_0 \oplus D_1$
- $Z_{32}yZ_{32}yZ_{21}^{(3)}xv \mapsto -\frac{1}{3}Z_{32}yZ_{31}zZ_{21}x\partial_{21}(v)$; with $v \in D_7 \otimes D_3 \oplus D_1$
- $Z_{32}yZ_{32}yZ_{21}^{(4)}xv \mapsto 0$; with $v \in D_8 \otimes D_2 \oplus D_1$
- $Z_{32}yZ_{32}yZ_{21}^{(5)}xv \mapsto -\frac{1}{10}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(3)}(v)$; with $v \in D_9 \otimes D_1 \oplus D_1$
- $Z_{32}yZ_{32}yZ_{21}^{(6)}xv \mapsto 0$ with $v \in D_{10} \otimes D_0 \oplus D_1$

- $Z_{32}^{(3)}yZ_{21}^{(4)}xZ_{21}xv \mapsto -\frac{1}{36}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(2)}\partial_{31}(v) -$

$\frac{1}{12}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(3)}\partial_{32}(v)$; with $v \in D_9 \otimes D_2 \oplus D_0$

- $Z_{32}^{(3)}yZ_{21}^{(5)}xZ_{21}xv \mapsto \frac{1}{6}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(3)}\partial_{31}(v) +$

$\frac{1}{30}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(4)}\partial_{32}(v)$; with $v \in D_{10} \otimes D_1 \oplus D_0$

- $Z_{32}^{(3)}yZ_{21}^{(4)}xZ_{21}^{(2)}xv \mapsto \frac{1}{12}Z_{32}yZ_{31}zZ_{21}x\partial_{32}\partial_{21}^{(4)}(v); v \in D_{10} \otimes D_1 \oplus D_0$
- $Z_{32}^{(3)}yZ_{21}^{(6)}xZ_{21}xv \mapsto \frac{-7}{60}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(4)}\partial_{31}(v)$; with

$v \in D_{11} \otimes D_0 \oplus D_0$

- $Z_{32}^{(3)}yZ_{21}^{(5)}xZ_{21}^{(2)}xv \mapsto 0$; with $v \in D_{11} \otimes D_0 \oplus D_0$
- $Z_{32}^{(3)}yZ_{21}^{(4)}xZ_{21}^{(3)}xv \mapsto \frac{-1}{6}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(4)}\partial_{31}(v)$; with

$v \in D_{11} \otimes D_0 \oplus D_0$

- $Z_{32}^{(3)}yZ_{32}yZ_{21}^{(4)}xv \mapsto \frac{-1}{6}Z_{32}yZ_{31}zZ_{21}x\partial_{21}\partial_{31}(v) -$

$\frac{1}{12}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(2)}\partial_{31}(v)$; with $v \in D_8 \otimes D_3 \oplus D_0$

- $Z_{32}^{(2)}yZ_{32}yZ_{21}^{(6)}xv \mapsto$

$\frac{-1}{60}(Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(3)}\partial_{31}(v) + Z_{32}yZ_{31}zZ_{21}x\partial_{31}\partial_{21}^{(4)}(v))$; with

$v \in D_{10} \otimes D_1 \oplus D_0$

- $Z_{32}^{(2)}yZ_{32}yZ_{21}^{(7)}xv \mapsto 0$; with $v \in D_{11} \otimes D_0 \oplus D_0$
- $Z_{32}yZ_{21}^{(2)}yZ_{21}^{(6)}xv \mapsto$

$\frac{-1}{12}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(3)}\partial_{31}(v) - \frac{1}{60}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(4)}\partial_{31}(v)$; with

$v \in D_{10} \otimes D_1 \oplus D_0$

- $Z_{32}yZ_{21}^{(2)}yZ_{21}^{(7)}xv \mapsto 0$; with $v \in D_{11} \otimes D_0 \oplus D_0$
- $Z_{32}yZ_{32}yZ_{21}yv \mapsto 0$; with $v \in D_4 \otimes D_7 \oplus D_0$
- $Z_{32}yZ_{31}zZ_{21}^{(2)}x(v) \mapsto \frac{1}{3}Z_{32}yZ_{31}zZ_{21}x\partial_{21}(v)$; with $v \in D_7 \otimes D_3 \oplus D_1$
- $Z_{32}yZ_{31}zZ_{21}^{(3)}x(v) \mapsto \frac{1}{3}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(2)}(v)$; with $v \in D_8 \otimes D_2 \oplus D_1$

- $Z_{32}yZ_{31}zZ_{21}^{(4)}x(v)\mapsto\frac{1}{10}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(3)}(v)$; with $v\in D_9\otimes D_1\oplus D_1$
- $Z_{32}yZ_{31}zZ_{21}^{(5)}x(v)\mapsto\frac{1}{5}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(4)}(v)$; with $v\in D_{10}\otimes D_0\oplus D_1$
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(2)}x(v)\mapsto\frac{1}{3}Z_{32}yZ_{31}zZ_{21}x\partial_{31}(v)$; with $v\in D_7\otimes D_4\oplus D_0$
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(3)}x(v)\mapsto$
$\frac{1}{12}(Z_{32}yZ_{31}zZ_{21}x\partial_{21}\partial_{31}(v)+Z_{32}yZ_{31}zZ_{21}x\partial_{32}\partial_{21}^{(2)}(v))$; with $v\in D_8\otimes D_3\oplus D_0$
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(4)}x(v)\mapsto$
$\frac{1}{9}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(2)}\partial_{31}(v)+\frac{1}{30}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(3)}\partial_{32}(v)$; with $v\in D_8\otimes D_3\oplus D_0$
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(5)}x(v)\mapsto\frac{1}{20}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(4)}\partial_{32}(v)$; with
$v\in D_{10}\otimes D_1\oplus D_0$
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(7)}x(v)\mapsto\frac{1}{15}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(4)}\partial_{32}(v)$; with
$v\in D_{11}\otimes D_0\oplus D_0$
- $Z_{32}yZ_{32}yZ_{21}z(v)\mapsto 0$; with $v\in D_5\otimes D_6\oplus D_0$
- $Z_{32}^{(2)}yZ_{32}yZ_{21}^{(5)}x(v)\mapsto\frac{1}{20}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(3)}\partial_{32}(v)$; with
$v\in D_9\otimes D_2\oplus D_0$

Again easily we can show that σ_3 which defined above satisfies the condition (3.3), and here we chose one of them as an example

$$(\delta_{B_3A_2}+\sigma_2\circ\delta_{B_3B_2})(Z_{32}^{(2)}yZ_{31}zZ_{21}^{(4)}x(v))=$$
$$=2\sigma_2(Z_{32}^{(3)}yZ_{21}^{(5)}x(v))+\sigma_2(Z_{32}^{(3)}yZ_{21}^{(5)}x\partial_{32}(v))+\sigma_2(Z_{32}^{(3)}yZ_{31}z\partial_{21}^{(2)}(v))=$$
$$=\frac{2}{9}Z_{32}yZ_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v)+\frac{1}{9}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)-\frac{1}{15}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)$$
$$+\frac{1}{30}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}(v)-\frac{1}{5}Z_{32}yZ_{31}zZ_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}(v)+\frac{1}{3}Z_{32}yZ_{31}z\partial_{21}^{(4)}\partial_{32}(v)$$
$$+\frac{1}{3}Z_{32}yZ_{31}z\partial_{21}^{(3)}\partial_{31}(v)=$$
$$\frac{2}{9}Z_{32}yZ_{21}^{(2)}x\partial_{21}\partial_{32}^{(2)}(v)+\frac{13}{90}Z_{32}yZ_{31}zZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)$$
$$+\frac{1}{15}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)+\frac{2}{15}Z_{32}yZ_{31}z\partial_{21}^{(4)}\partial_{32}(v)+\frac{1}{3}Z_{32}yZ_{31}z\partial_{21}^{(3)}\partial_{31}(v)$$

and

$$(\delta_{A_3A_2}+\sigma_2\circ\delta_{A_3B_2})\,(\frac{1}{9}Z_{32}y\,Z_{31}z\,Z_{21}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)$$
$$+\frac{1}{30}Z_{32}y\,Z_{31}z\,Z_{21}x\partial_{21}^{(3)}\partial_{32}\partial_{32}(v))$$
$$=\frac{1}{9}Z_{32}y\,Z_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)+\frac{2}{9}Z_{32}y\,Z_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}$$
$$+\frac{1}{3}Z_{32}y\,Z_{31}z\,\partial_{21}^{(3)}\partial_{31}(v)+\frac{1}{30}Z_{32}y\,Z_{21}^{(2)}x\partial_{32}\partial_{21}^{(3)}\partial_{32}(v)+\frac{2}{15}Z_{32}y\,Z_{31}z\,\partial_{21}^{(4)}\partial_{32}(v)$$
$$=\frac{1}{9}Z_{32}y\,Z_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)+\frac{2}{9}Z_{32}y\,Z_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v)$$
$$+\frac{1}{3}Z_{32}y\,Z_{31}z\,\partial_{21}^{(3)}\partial_{31}(v)+\frac{1}{15}Z_{32}y\,Z_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)$$
$$\frac{1}{30}Z_{32}y\,Z_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)+\frac{2}{15}Z_{32}y\,Z_{31}z\,\partial_{21}^{(4)}\partial_{32}(v)$$
$$=\frac{13}{90}Z_{32}y\,Z_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)+\frac{2}{9}Z_{32}y\,Z_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v)$$
$$+\frac{1}{3}Z_{32}y\,Z_{31}z\,\partial_{21}^{(3)}\partial_{31}(v)+\frac{1}{15}Z_{32}y\,Z_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)+\frac{2}{15}Z_{32}y\,Z_{31}z\,\partial_{21}^{(4)}\partial_{32}(v).\square$$

So from all we have done above we have the complex

$$0\longrightarrow A_3\xrightarrow{\partial_3}A_2\xrightarrow{\partial_2}A_1\xrightarrow{\partial_1}A_0\quad\ldots(3.4)$$

where ∂_i defined as follows :

- $\partial_1(Z_{21}x(v))=\partial_{21}(v)$; with $v\in D_5\otimes D_3\oplus D_3$
- $\partial_1(Z_{32}y(v))=\partial_{32}(v)$; with $v\in D_4\otimes D_5\oplus D_2$
- $\partial_2(Z_{32}y\,Z_{21}^{(2)}x(v))=\frac{1}{2}Z_{21}x\partial_{21}\partial_{32}(v)+Z_{21}x\partial_{31}(v)-Z_{32}y\partial_{21}^{(2)}(v)$; with $v\in D_6\otimes D_3\oplus D_2$.
- $\partial_2(Z_{32}y\,Z_{31}z(v))=\frac{1}{2}Z_{32}y\partial_{32}\partial_{21}(v)-Z_{21}x\partial_{32}^{(2)}(v)-Z_{32}y\partial_{32}^{(2)}(v)$; with $v\in D_5\otimes D_5\oplus D_1$.

and the map ∂_3 is defined as

- $\partial_3(Z_{32}yZ_{31}z\,Z_{21}x(v))=Z_{32}yZ_{21}^{(2)}x\partial_{32}(v)+Z_{32}yZ_{31}z\,\partial_{21}(v)$; with $v\in D_6\otimes D_4\oplus D_1$.

Proposition (3.4) [1],[2] :

The complex

$$0 \longrightarrow A_3 \xrightarrow{\partial_3} A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \longrightarrow K_{(4,4,3)} \text{ is exact.}\square$$

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اختزال تحلل مقاس وايل من صيغة المميز الحر الى تحلل لاسكو في حالة (4,4,3)

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استلم البحث في:8 كانون الثاني 2012 قبل البحث في: 22 نيسان 2012

الخلاصة

في هذا البحث ندرس العلاقة بين تحلل مقاس وايل ($K_{(4,4,3)}F$) في صيغة المميز الحر والتحلل للاسكو في صيغة المميز صفر و بدقة اكثر سوف نحصل على تحلل لاسكو لـ $K_{(4,4,3)}F$ في صيغة المميز صفر تطبيقا لتحلل $K_{(4,4,3)}F$ في صيغة المميز الحر.

الكلمات المفتاحية : تحلل، مقاس وايل، تحلل لاسكو، تقسيم القوة، المميز الحر.