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Strongly (Comletely) Hollow Submodules I

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Abstract

Let R be a commutative ring with unity and let M be an R-module. In this paper we study strongly (completely) hollow submodules and quasi-hollow submodules. We investigate the basic properties of these submodules and the relationships between them. Also we study the be behavior of these submodules under certain class of modules such as compultiplication, distributive, multiplication and scalar modules. In part II we shall continue the study of these submodules.

Key Words: Strongly (completely)-hollow submodules, distributive modules, multiplication (comultiplication) modules, scalar modules.

Introduction

Throughout this paper, all rings are commutative rings with identity elements, and all modules are unital modules. In this article we study strongly (completely) hollow submodules which are introduced in [1], also we introduce quasi-hollow submodules. In section one of this paper we give the basic properties of these submodules. Also we give some results under the class of distributive modules and compultiplication modules. In section two, we investigate some properties of strongly, completely and quasi-hollow submodules under the class of multiplication modules. In section three we introduce some properties of strongly (completely) and quasi-hollow submodules under certain class of modules.

1- Strongly (Completely) Hollow and Quasi-Hollow Submodules

We begin this section with the following:

Definition: [1, 4.2]

Let $0 \neq L \leq M$, then L is called a strongly hollow submodule (briefly, SH-submodule) if for every $L_1, L_2 \leq M$ with $L \leq L_1 + L_2$ implies $L \leq L_1$ or $L \leq L_2$, we say that an R-module M is a strongly-hollow module if M is a strongly hollow submodule of itself.

Remark:

Let $0 \neq L \leq M$, L is a SH-submodule if for each $L_1, \dots, L_n \leq M$ with $L \leq L_1 + L_2 + \dots + L_n$, implies $L \leq L_1$ or $L \leq L_2 \dots$ or $L \leq L_n$.

Definition: [1, 4.2]

Let $0 \neq L \leq M$, then L is called a completely hollow submodule (briefly, CH-submodule) if for any collection $\{L_\lambda\}_{\lambda\in\Lambda}$ of R-submodules of M with $L = \sum_{\lambda\in\Lambda} L_\lambda$, implies $L = L_\lambda$ for some $\lambda\in\Lambda$.

We say that an R-module M is completely hollow (briefly, CH-module) if M is completely hollow of itself.

Remarks and Examples:

The Z as Z-module is not SH, not CH, and every submodule is not SH, not CH.

1. Z_6 as Z-module is not SH, and every nonzero proper submodule is SH.

2. Q as Z-module is not SH, since there exist two proper submodules A, B of Q such that Q = A + B see [2, p.187, Exc.6(b)].

3. Let M be an R-module, and $N \le L \le M$.

If L is SH then N need not be SH-submodule.

For example, $<\overline{2}>$ is SH (CH)-submodule of Z_4 as Z-module. But $<\overline{0}>$ is not SH (not CH).

4. Let M be an R-module, and $0 \ne L \le W \le M$.

If L is SH-submodule, then W need not be SH-submodule.

For example $<\overline{6}>$ is SH(CH)-submodule of Z_{48} as Z-module. But $<\overline{2}>$ is not SH (not CH), since $<\overline{2}> \subseteq <\overline{8}> + <\overline{6}>$, and $<\overline{2}> \not\subseteq <\overline{8}>$, $<\overline{2}> \not\subseteq <\overline{6}>$.

5. Let M be an R-module, and $L_1, L_2 \le M$. If L_1 and L_2 are SH-submodule, then $L_1 + L_2$ need not be SH.

For example: In Z_{12} as Z-module, $<\overline{3}>, <\overline{4}>$ are SH-submodules of Z_{12}. But $<\overline{3}> + <\overline{4}> = Z_{12}$ is not SH.

6. If M is a chained R-module, and $0 \ne N \le M$. Then N is SH-submodule, where M is a chained module if the Lattic of submodules are linearly ordered by inclusion see [3].

Proof:

Let $0 \ne N \le M$. Assume $N \subseteq N_1 + N_2$ where $N_1, N_2 \le M$. Since M is chained, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$

If $N_1 \subseteq N_2$, then $N_1 + N_2 = N_2$, so $N \subseteq N_2$.

If $N_2 \subseteq N_1$, then $N_1 + N_2 = N_1$, so $N \subseteq N_1$.

Thus N is SH-submodule.

7. Every simple R-module M is SH and CH.

8. Every simple submodule N of an R-module is CH-submodule.

9. Every CH-module is SH-module.

10. The concept SH-submodule and CH-submodule are independent

For examples:

(a) The Z-module Z_{p^∞} is SH-submodule of itself; that is Z_{p^∞} is SH-module by Remark 1.4 (7), Z_{p^∞} is not CH-module. Since $Z_{p^\infty} = \sum_{i \in Z_+} <\frac{1}{p^i} + Z>$, and $Z_{p^\infty} \ne <\frac{1}{p^i} + Z>$ for any $i \in Z_+$.

(b) Let M be the vector space $\mathbb{R}^2$ over $\mathbb{R}$. Let $N = \mathbb{R}_{(1,0)}$. N is simple submodule of M. Since dim N = 1. So by Remark 1.4 (9), N is CH. On the other hand, $N \subseteq \mathbb{R}_{(1,1)} + \mathbb{R}_{(1,-1)} = \mathbb{R}^2 = M$, and $N \not\subseteq \mathbb{R}_{(1,1)}$, $N \not\subseteq \mathbb{R}_{(1,-1)}$. That is N is not SH-submodule.

As we have seen by Example 1.4 (11) (b), simple submodule need not be SH. However under the class of distributive (or comultiplication) modules, every simple submodule is SH. Before proving this result, recall that the following definitions

An R-module M is called distributive if the Lattic of its submodules is distributive, that is $L \cap (N + K) = (L \cap N) + (L \cap K)$.

Equivalently, $L + (N \cap K) = (L + N) \cap (L + K)$ for all submodules L, N, and K of M see [4].

An R-module M is called comultiplication if every $L \le M$ is of the form $L = (O \underset{M}{:} I) = \underset{M}{\text{ann}}\, I$ for some $I \le R$. Equivalently, $L = (O \underset{M}{:} (O \underset{R}{:} L)) = \underset{M}{\text{ann}}\ \underset{R}{\text{ann}}\, L$, see [5].

where $(O \underset{M}{:} I) = \{m \in M: Im = (0)\}$, $(O \underset{R}{:} L) = \{r \in R: rL = (0)\}$.

Examples:

1. Z_{p^∞} as Z-module is comultiplication, since for each $L \leq Z_{p^\infty}$ $L = <\frac{1}{p^i} + Z>$, then $\underset{Z_{p^\infty}}{\text{ann}}\ \underset{Z}{\text{ann}}\, L = L$ for some $i \in Z_+$.

2. Z as Z-module is not comultiplication, since if L = 3Z, then $\underset{Z}{\text{ann}}\ \underset{Z}{\text{ann}}\, 3Z = Z \neq 3Z$.

3. Z_n as Z-module is comultiplication.

Proof:

Let $L \leq M$. Then $L = <\bar{m}>$ and m/n, that is n = mk for some $k \in Z$. Hence $\underset{Z}{\text{ann}} <\bar{m}> = <k>$ and $\underset{Z}{\text{ann}} <k> = <\bar{m}> = L$. Thus $L = \underset{Z_n}{\text{ann}}\ \underset{Z}{\text{ann}}\, L$.

Recall that a non-zero submodule N of an R-module M is said to be second submodule of M if for each $r \in R$, the homothety r^* on N is either zero or surjective. Equivalently, rN = <0> or rN = N for each $r \in R$, see [6].
where the homothety r^* is an R-endomorphism on N, means $r^*(x) = rx$ for each $x \in N$.

A submodule N of an R-module M is said to be strongly irreducible (briefly, SI-submodule) if for any $L_1, L_2 \leq M$, $L_1 \cap L_2 \subseteq N$, then $L_1 \subseteq N$ or $L_2 \subseteq N$, see [7].

Examples:

1. 6Z is not SI-submodule of Z as Z-module since $6Z \supseteq 2Z \cap 3Z$, but $6Z \not\supseteq 2Z$, $6Z \not\supseteq 3Z$.

2. It is clear that every submodule of chained module is SI.

We state the following proposition which is needed in the next two results.

Proposition:

Let M be a comultiplication R-module, and $N \leq M$ such that $\underset{R}{\text{ann}}\, N$ is prime ideal. Then N is a SH-submodule.

Proof:

Let $N \subseteq L_1 + L_2$, where $L_1, L_2 \leq M$. Since M is comultiplication, $L_1 = \underset{M}{\text{ann}}\, I_1$, $L_2 = \underset{M}{\text{ann}}\, I_2$ for some ideals I_1 and I_2 of R. Then $N \subseteq \underset{M}{\text{ann}}\, I_1 + \underset{M}{\text{ann}}\, I_2 \subseteq \underset{M}{\text{ann}}\, (I_1 \cap I_2)$, that is $N \subseteq \underset{M}{\text{ann}}\, (I_1 \cap I_2)$. So $\underset{R}{\text{ann}}\, N \supseteq \underset{R}{\text{ann}}\ \underset{M}{\text{ann}}\, (I_1 \cap I_2) \supseteq I_1 \cap I_2$. But $\underset{R}{\text{ann}}\, N$ is prime so $\underset{R}{\text{ann}}\, N$ is SI-ideal, hence $\underset{R}{\text{ann}}\, N \supseteq I_1$ or $\underset{R}{\text{ann}}\, N \supseteq I_2$. Then $\underset{M}{\text{ann}}\ \underset{R}{\text{ann}}\, N \subseteq \underset{M}{\text{ann}}\, I_1 = L_1$ or $\underset{M}{\text{ann}}\ \underset{R}{\text{ann}}\, N \subseteq \underset{M}{\text{ann}}\, I_2 = L_2$. So $N \subseteq L_1$ or $N \subseteq L_2$, that is N is SH.

The following result is given in [1]. However we get it directly by Proposition 1.7.

Corollary:

Let M be a comultiplication R-module, and $N \leq M$. Then

1. N is a second submodule implies N is SH.

2. N is a finitely generated second submodule, implies N is CH.

Proof:

(1) Since N is second, then $\underset{R}{\text{ann}}\, N$ is a prime ideal by [6]. Hence the result is obtained by Proposition 1.7.

(2) By part (1) N is SH. But N is finitely generated, so $N = \sum_{i=1}^{n} Rx_i$ for some $x_1, \dots, x_n$. Hence $N \subseteq Rx_i$ for some i = 1, …, n. But $Rx_i \subseteq N$. Thus $N = Rx_i$.

Corollary:

Let M be a comultiplication R-module, and let N be a simple submodule. Then N is SH.

Proof:

It is clear that every simple submodule is second, hence the result follows by corollary 1.8 (1).

Recall that an R-module M is said to be prime if $\underset{R}{\text{ann}}\, M = \underset{R}{\text{ann}}\, N$ for every non-zero submodule N of M., see [8].

If M is a prime R-module, then $\underset{R}{\text{ann}}\, M$ is prime by [8].

An R-module M is called a quasi-prime if $\underset{R}{\text{ann}}\, N$ is a prime for each non-zero submodule N of M, see [9, Definition 1.2.1].

Notice that every prime R-module M is quasi-prime by [9, Remark 1.2.2].

Corollary:

Let M be a comultiplication prime (or quasi-prime) R-module. Then every non-zero submodule is SH.

Proof:

Since M is prime (or quasi-prime) implies $\underset{R}{\text{ann}}\, N$ is prime ideal for each non-zero submodule N of M. Hence the result follows from Proposition 1.7.

Proposition:

Let M be a distributive R-module, and $<0> \neq N \leq M$. If N is a simple submodule of M, then N is SH.

Proof:

Assume N is simple, $N \leq L_1 + L_2$ where $L_1, L_2 \leq M$. Hence

$N = N \cap (L_1 + L_2)$

$\quad = (N \cap L_1) + (N \cap L_2)$, since M is distributive.

Then ($N \cap L_1 = <0>$ or $N \cap L_1 = N$) and ($N \cap L_2 = <0>$ or $N \cap L_2 = N$). But $N \neq 0$. So we have only three possible cases

(1) $N \cap L_1 = <0>$, $N \subseteq L_2$.

(2) $N \cap L_2 = <0>$, $N \subseteq L_1$.

(3) $N \subseteq L_1$, $N \subseteq L_2$.

Thus either $N \subseteq L_1$ or $N \subseteq L_2$; that is N is SH.

Remark:

The condition M is distributive or comultiplication is necessary condition in Proposition 1.11 and Corollary 1.9.

As we have seen in Remark 1.4(11)(b), $N = \mathbb{R}_{(1,0)}$ and N is simple but not SH. Moreover the vector space $\mathbb{R}^2$ over $\mathbb{R}$ is not distributive since $\mathbb{R}^2 = \mathbb{R}_{(1,1)} + \mathbb{R}_{(1,-1)}$ and $N \cap \mathbb{R}^2 = N$, but $(N \cap \mathbb{R}_{(1,1)}) + (N \cap \mathbb{R}_{(1,-1)}) = \{(0,0)\}$. Thus $\mathbb{R}^2$ is not distributive.

Also $\mathbb{R}^2$ is not comultiplication R-module. For if $L = \mathbb{R}_{(1,1)}$, then $\underset{\mathbb{R}}{\text{ann}}\, L = \{0\}$ and $\underset{\mathbb{R}^2}{\text{ann}}\{0\} = \mathbb{R}^2$, thus $L \neq \underset{\mathbb{R}^2}{\text{ann}}\, \underset{\mathbb{R}}{\text{ann}}\, L$.

Now we introduce the following concept.

Definition:

Let $<0> \neq L \leq M$, L is called a quasi-hollow submodule (briefly qH-submodule) if for each $L_1, L_2 \leq M$ with $L = L_1 + L_2$, then $L = L_1$ or $L = L_2$.

An R-module M is said a quasi-hollow module if M is a quasi-hollow submodule.

Remark:

Let <0> ≠ L ≤ M, L is a quasi-hollow submodule if for each $L_1, \dots, L_n$ with $L = L_1 + \dots + L_n$, then $L = L_1$ or … or $L = L_n$.

Remarks and Examples:

1. It is clear that every CH-submodule is qH-submodule. The converse is not true. For example the Z-module Z_{p^∞} is qH-module (qH-submodule of itself) since there is no $L_1 \subsetneqq M$ and $L_2 \subsetneqq M$ such that $Z_{p^\infty} = L_1 + L_2$. But by Remark 1.4(11)(b) Z_{p^∞} is not CH.

2. Every simple submodule of an R-module is qH-submodule.

3. Every SH-submodule is qH-submodule.

The converse is not true in general, for example in the vector space $\mathbb{R}^2$ over $\mathbb{R}$, $N = \mathbb{R}_{(1,0)}$ is simple submodule, so by Remark 1.15(2), N is qH, but it is not SH by Remark 1.4(11)(b).

4. If M is chained, then every submodule is qH.

Proof:

It follows by Remark 1.4(7) and Remark 1.15(3).

5. Let M be an R-module. Then M is a qH-module if and only if M is SH if and only if M is hollow.

where M is hollow if every proper submodule N of M is small.

That is there is no proper submodule W of M such that N + W = M.

Equivalently, for every submodules N, W such that $N \subsetneqq M$, $W \subsetneqq M$ implies $N + W \subsetneqq M$.

6. Let M be CH (qH or SH) R-module, then there is no submodules N, W of M such that $M = N \oplus W$.

7. Consider Z_{48} as Z-module. Each of $<\bar{2}>, <\bar{4}>, <\bar{8}>$ and Z_{48} is not qH, not SH. Each of $<\bar{3}>, <\bar{6}>, <\overline{12}>, <\overline{24}>$ is qH and SH.

8. Consider $M = Z_4 \oplus Z_2$ as Z-module.

Each of $<\bar{0}> \oplus Z_2, Z_4 \oplus <\bar{0}>, <\bar{2}> \oplus <\bar{0}>$ is qH and SH, and each of $Z_2 \oplus Z_4, <\bar{2}> \oplus Z_2$ is not qH, not SH.

9. Let <0> ≠ L ≤ M as R-module. Let N ≤ L. If L is qH-submodule, then N need not be qH. For example, Z-module Z_{48} where $<\bar{3}>$ is qH, but $<\bar{0}>$ is not qH.

10. Let <0> ≠ L ≤ W ≤ M as R-module. If L is qH, then W need not be qH. For example, $M = Z_4 \oplus Z_2$ as Z-module, where $<\bar{0}> \oplus Z_2$ is qH and $<\bar{0}> \oplus Z_2 \subseteq <\bar{2}> \oplus Z_2$. But $<\bar{2}> \oplus Z_2$ is not qH.

11. If L_1, L_2 are qH of an R-module M, then $L_1 + L_2$ need not be qH. For example, $L_1 = <\bar{0}> \oplus Z_2$, $L_2 = Z_4 \oplus <\bar{0}>$ are qH of $M = Z_4 \oplus Z_2$ as Z-module. But $L_1 + L_2 = M$ is not qH.

12. Let R be a ring. If A and B are qH(SH)-ideals. Then AB need not be qH(SH)-ideals of R. For example $<\bar{2}>$ and $<\bar{3}>$ are qH(SH)-ideals of the ring Z_6. But $<\bar{2}> \cdot <\bar{3}> = <\bar{0}>$ is not SH, not qH.

Now we find that under the class of distributive of modules, the concepts, qH-submodules and SH-submodules are equivalent, as the following proposition shows:

Proposition:

Let M be distributive R-module, and $0 \neq N \subsetneqq M$. Then N is SH-submodule if and only if N is qH-submodule.

Proof:

(⇒) Clear by Remark 1.15(3).

(⇐) Assume N is qH-submodule. Let $N \subseteq L_1 + L_2$ where $L_1, L_2 \leq M$. Then $N = N \cap (L_1 + L_2)$, so $N = (N \cap L_1) + (N \cap L_2)$, since M is distributive. Then $N = N \cap L_1$ or $N = N \cap L_2$ since N is qH. It follows that either $N \subseteq L_1$ or $N \subseteq L_2$. Hence N is a SH-submodule.

Corollary:

Let M be distributive R-module, and $<0> \neq N \subsetneqq M$. If N is CH-submodule, then N is SH.

Proof:

It follows by Remark 1.15(1) and previous proposition.

Remark:

Let M be an R-module, $N \subseteq K \subseteq M$. If N is SH(qH)-submodule in M, then N is SH(qH) in K.

Proof: It is clear

The converse of this remark is true under the class of distributive module as follows:

Proposition:

Let M be a distributive R-module. Let $N \subseteq K \subseteq M$. Then N is SH(qH)-submodule in M if and only if N is SH(qH) in K.

Proof:

(⇒) It follows by previous remark.

(⇐) Assume N is SH-submodule in K. Let $N \subseteq L_1 + L_2$ where $L_1, L_2 \leq M$. Since $N \subseteq K$ then $N = N \cap K \subseteq (L_1 + L_2) \cap K$

$= (L_1 \cap K) + (L_2 \cap K)$, since M is distributive

So $N \subseteq (L_1 \cap K) + (L_2 \cap K)$. Then $N \subseteq L_1 \cap K$ or $N \subseteq L_2 \cap K$, since N is SH in K.

Then $N \subseteq L_1$ or $N \subseteq L_2$. Thus N is SH in M.

By a similar proof, if N is qH in K, then N is qH in M.

Now we turn our attention to image and inverse image of SH, qH and CH-submodules.

Proposition:

Let M and M' be R-modules and N be a SH-submodule of M. If $f : M \longrightarrow M'$ be an R-epimorphism, then $f(N)$ is SH-submodule of M'.

Proof:

Let $f(N) \subseteq L_1 + L_2$ where $L_1, L_2 \leq M'$. Then $N \subseteq f^{-1}f(N) \subseteq f^{-1}(L_1 + L_2)$. But $f^{-1}(L_1 + L_2) = f^{-1}(L_1) + f^{-1}(L_2)$ see [2,3.1.10(c)], so $N \subseteq f^{-1}(L_1) + f^{-1}(L_2)$, then $N \subseteq f^{-1}(L_1)$ or $N \subseteq f^{-1}(L_2)$. It follows $f(N) \subseteq f f^{-1}(L_1) = L_1$ or $f(N) \subseteq f f^{-1}(L_2) = L_2$. Hence $f(N) \subseteq L_1$ or $f(N) \subseteq L_2$.

The condition f is an epimorphism is necessary in Proposition 1.20, for example, Let $f : Z_{12} \longrightarrow Z_{12}$, $f(x) = 4x$ fo each $x \in Z_{12}$, where Z_{12} considered as Z-module. It is clear that f is not epimorphism. Let $N = <\overline{3}>$, N is a SH submodule of Z_{12}. But $f(N) = <\overline{0}>$ is not SH.

1.21 Corollary:

Let N be a SH-submodule of an R-module M. Let $L \subsetneqq N$, then N/L is SH-submodule of M/L.

Corollary:

Let $M \cong M'$ be R-module, if $N \leq M$. Then N is a SH-submodule of M iff $f(N)$ is a SH-submodule of M'.

Proposition:

Let M and M' be R-modules and $f : M \longrightarrow M'$ be an isomorphism, Let $<0> \neq N \leq M$. If N is qH(CH)-sbmodule of M, then $f(N)$ is qH (CH)-submodule of M'.

Proof:

If N is qH-submodule of M. Assume $f(N) = W_1 + W_2$ for some $W_1, W_2 \leq M'$. Since f is isomorphism, so $W_1 = f(L_1)$, $W_2 = f(L_2)$ for some $L_1, L_2 \leq M$. Thus $f(N) = f(L_1) + f(L_2)$. But $f(L_1 + L_2) = f(L_1) + f(L_2)$, see [2, 3.1.10(a)]. Then $f(N) = f(L_1 + L_2)$. Since f is monomorphism, we get $N = L_1 + L_2$. It follows that $N = L_1$ or $N = L_2$. Hence $f(N) = f(L_1) = W_1$ or $f(N) = f(L_2) = W_2$. By a similar proof, N is CH-submodule of M implies $f(N)$ is CH of M'.

Proposition:

Let $f: M \longrightarrow M'$ be an isomorphism R-module. If K is SH(qH or CH)-submodule of M', then $f^{-1}(K)$ is SH(qH or CH)-submodule of M.

Proof:

Assume K is SH in M'. Let $f^{-1}(K) \subseteq L_1 + L_2$ where $L_1, L_2 \leq M$. Then $f f^{-1}(K) \subseteq f(L_1 + L_2) = f(L_1) + f(L_2)$, see [2,3.1.10(a)]. Since f is epimorphism $K = f f^{-1}(K) \subseteq f(L_1) + f(L_2)$. So $K \subseteq f(L_1)$ or $K \subseteq f(L_2)$. Thus $f^{-1}(K) \subseteq f f^{-1}(L_1) = L_1$ or $f^{-1}(K) \subseteq f f^{-1}(L_2) = L_2$. Since f is monomorphism. Hence $f^{-1}(K) \subseteq L_1$ or $f^{-1}(K) \subseteq L_2$.

By a similar proof, K is qH(CH) of M', then $f^{-1}(K)$ is qH(CH).

The condition that f is an isomorphism is necessary in Proposition 1.24. For example, Consider the Z-module Z and let $\pi: Z \longrightarrow Z/\langle 4\rangle \simeq Z_4$ be the natural projection. Let $K = \langle \bar{2} \rangle \subseteq Z_4$, K is SH(qH or CH) of Z_4. But $\pi^{-1}(k) = 2Z$ is not SH (not qH, not CH) in Z.

Now we give the next result of this section.

Proposition:

Let M_1, M_2 be R-modules. Let $M = M_1 \oplus M_2$, and let $\langle 0\rangle \neq K \subseteq M_1 \oplus M_2$. If $K = N \oplus W$ for some $N \leq M_1$, $W \leq M_2$ such that K is SH(qH)-submodule. Then N is SH(qH) of M_1 and W is SH(qH) of M_2.

Proof:

Assume $K = N \oplus W$, K is a SH submodule in M, $K = (N \oplus \langle 0\rangle) + (\langle 0\rangle \oplus W)$, so $K = N \oplus \langle 0\rangle$ or $K = \langle 0\rangle \oplus W$ since K is SH of M. If $K = N \oplus \langle 0\rangle$. We claim that N is SH of M_1. Assume $N \subseteq L_1 + L_2$ where $L_1, L_2 \leq M_1$. So $K \subseteq (L_1 + L_2) \oplus \langle 0\rangle$.

Then $K \subseteq (L_1 \oplus \langle 0\rangle) + (L_2 \oplus \langle 0\rangle)$. Then $K \subseteq L_1 \oplus \langle 0\rangle$ or $K \subseteq L_2 \oplus \langle 0\rangle$. Thus $N \oplus \langle 0\rangle \subseteq L_1 \oplus \langle 0\rangle$ or $N \oplus \langle 0\rangle \subseteq L_2 \oplus \langle 0\rangle$. Hence $N \leq L_1$ or $N \leq L_2$. Then N is SH of M_1.

Similarly, if $K = \langle 0\rangle \oplus W$, then W is SH of M_2.

By a similar proof, if K is qH, then N, W are qH in M_1, M_2 respectively.

The converse of Proposition 1.25 is not true in general.

For example in Z-module $M = Z_4 \oplus Z_6$. If $K = \langle \bar{2} \rangle \oplus \langle \bar{3} \rangle$, then K is not SH (not qH). But $\langle \bar{2} \rangle$ is SH(qH) in Z_4 and $\langle \bar{3} \rangle$ is SH(qH) in Z_6.

2- SH and qH(CH)-Submodules and Multiplication Modules

In this section, we introduce some properties of SH and qH(CH) submodles in the class of multiplication modules.

Recall that an R-module M is called multiplication if every $N \leq M$, N is of the form $N = IM$ for some ideal $I \leq R$. Equivalently, $N = (N \underset{R}{:} M) \cdot M$, where $(N \underset{R}{:} M) = \{r \in R, rM \subseteq N\}$, see [4].

Proposition:

Let M be a faithful finitely generated multiplication R-module, $N \leq M$. Then the following statements are equivalent:

(1) N is SH(qH)-submodule.

(2) $(N :_R M)$ is SH(qH)-ideal.

(3) $N = IM$, I is SH(qH)-ideal for some $<0> \neq I \leq R$.

Proof:

(1) ⇒ (2) Assume N is a SH-submodule of M, and let $(N :_R M) \subseteq I_1 + I_2$ where I_1, I_2 are ideals of R. Then $(N :_R M)\cdot M \subseteq (I_1 + I_2)\cdot M$. But $(I_1 + I_2)\cdot M = I_1M + I_2M$ since M is finitely generated. It follows $(N :_R M)\cdot M \subseteq I_1M + I_2M$. But $(N :_R M)\cdot M = N$ since M is multiplication. Then $N \subseteq I_1M + I_2M$, so either $N \subseteq I_1M$ or $N \subseteq I_2M$, that is $(N :_R M)\cdot M \subseteq I_1M$ or $(N :_R M)\cdot M \subseteq I_2M$. Thus $(N :_R M) \subseteq I_1$ or $(N :_R M) \subseteq I_2$ by [10, Theorem 3.1]. Hence $(N :_R M)$ is a SH-ideal of R.

By a similar proof, $(N :_R M)$ is qH-ideal if N is qH.

(2) ⇒ (3) Assume $(N :_R M)$ is SH-ideal. Put $I = (N :_R M)$. Since M is multiplication, then $N = (N :_R M)\cdot M$. Hence $N = IM$, and I is a SH-ideal. Similarly I is a qH-ideal.

(3) ⇒ (1) Assume that $N = IM$ for some SH-ideal I of R, and let $N \subseteq L_1+L_2$ where $L_1, L_2 \leq M$. But $L_1 = I_1M$, $L_2 = I_2M$ since M is multiplication for some ideals I_1, I_2 of R. So $IM \subseteq I_1M+I_2M \subseteq (I_1 + I_2)\cdot M$, since M is finitely generated. Then $IM \subseteq (I_1 + I_2)\cdot M$, so $I \subseteq I_1 + I_2$ by [10, Theorem 3.1].
So that either $I \subseteq I_1$ or $I \subseteq I_2$, which implies $IM \subseteq I_1M$ or $IM \subseteq I_2M$. Hence $N \subseteq L_1$ or $N \subseteq L_2$. Thus N is SH.
Similarly N is qH.

The condition M is faithful is necessary in Proposition 2.1. Fot example, the Z-module Z_6 is finitely generated multiplication, but not faithful, let $N = <\overline{2}>$, N is SH of Z_6,but $(N :_Z M) = (<\overline{2}> :_Z Z_6) = 2Z$ is not SH of Z.

Corollary:

Let M be a finitely generated faithful multiplication R-module. Then every non-zero submodule of M is SH(qH) if and only if every non-zero ideal of R is SH(qH).

Proposition:

Let M be a faithful finitely generated multiplication R-module. Then R satisfies acc(dcc) on SH-ideals if and only if M satisfies acc(dcc) on SH-submodules.

Proof:

⇒ We take the case of acc.
Let $L_1 \subseteq L_2 \ldots$ be an ascending chain of SH-submodules of M. Since L_i is SH-submodule, then $(L_i :_R M)$ is SH-ideal for each i = 1,2,… by Proposition 2.1, and $(L_1 :_R M) \subseteq (L_2 :_R M) \subseteq \ldots$ by [10,Theorem 3.1]. But R satisfies acc on any ascending chain of SH-ideals. So ther exists n $\in Z_+$ such that $(L_n :_R M) = (L_{n+1} :_R M) = \ldots.$ Then $(L_n :_R M)\cdot M = (L_{n+1} :_R M)\cdot M = \ldots..$ Thus $L_n = L_{n+1} = \ldots$ for some $n \in Z_+$. Hence M satisfies acc on SH-submodules.
⇐ The proof is similar.

SH, qH(CH)-Submodules and Other Related Concepts

Recall that an R-module M is called scalar if for each $f \in End_R(M)$, there exists $r \in R$ such that $f(x) = rx$ for all $x \in M$, see [11].

Proposition:

Let M be a scalar R-module and R is SH-ring, then $End_R(M)$ is SH-ring.

Proof:

Since M is a scalar R-module, then $End_R(M) \simeq R/\underset{R}{ann}M$, see [12, Lemma 3.6.1]. Since R is SH-ring, then $R/\underset{R}{ann}M$ is SH-ring by Corollary 1.21. Thus $End_R(M)$ is SH-ring by Corollary 1.22.

Corollary:

Let M be a finitely generated multiplication module over SH-ring. Then $End_R(M)$ is SH-ring.

Proof:

Since M is finitely generated multiplication, then M is scalar, see [11]. Hence the result is obtained by Proposition 3.1.

Next we shall prove in the class of comultiplication prime modules, every submodule of M is SH (qH)-module. But first we prove the following proposition and lemma.

Proposition:

Let M be a comultiplication R-module. If M is prime (quasi-prime or second). Then M is hollow.

Proof:

Since M is prime (quasi-prime or second), then $\underset{R}{ann}M$ is prime, see [8], [9], [6]. And $End_R(M)$ is domain, see [5, Corollary 3.21]. Hence M is hollow, see [5, Theorem 3.24].

Lemma:

Let M be a comultiplication R-module and $N \leq M$. Then N is a comultiplication R-module.

Proof:

Let $W \leq N$. So W is a submodule of M. Then there exists $I \leq R$ such that $W = \underset{M}{ann}I$. We claim that $W = \underset{N}{ann}I$. To prove our assertion. Let $m \in W$ (so, $m \in N$). Hence $mI = 0$, so $m \in \underset{N}{ann}I$. Now let $m \in \underset{N}{ann}I$, so $m \in N$ and $mI = 0$, $m \in M$. Thus $m \in \underset{M}{ann}I$. Then $w = \underset{N}{ann}I$. Henc N is comultiplication.

Theorem:

Let M be a comultiplication prime R-module. Then every non-zero submodule of M is a SH(qH) R-module.

Proof:

Since M is comultiplication prime, then by Proposition 3.3, M is hollow. Let N be a non-zero submodule of M, then N is comultiplication by Lemma 3.4. But M is prime implies N is a prime R-module. Thus N is a hollow R-module by Proposition 3.3. Hence N is qH(SH)-R-module, see Remark 1.15(5).

Corollary:

Let M be a comultiplication prime R-module. Then every non-zero submodule of M is qH-sumodule of M.

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المقاسات الجزئية المجوفة (التامة) بقوة I

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الخلاصة

لتكن R حلقة ابدالية ذا محايد. وليكن M مقاساً على R . في هذا البحث درسنا المفاهيم: المقاسات الجزئية المجوفة بقوة (التامة) والمقاسات الجزئية شبه المجوفة وقدمنا الخواص المتعلقة بهم والعلاقات فيما بينهم. كذلك درسنا سلوك هذه المقاسات الجزئية في أصناف معينة من المقاسات ، مثل المقاسات التوزيعية، والمقاسات الجدائية المضادة، والمقاسات الجدائية والمقاسات القياسية.

الكلمات المفتاحية: المقاسات الجزئية المجوفة (التامة) بقوة، المقاسات التوزيعية، المقاسات الجدائية (الجدائية المضادة)، المقاسات القياسية.