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On Fuzzy Groups and Group Homomorphism

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Abstract

In this paper, we study the effect of group homomorphism on the chain of level subgroups of fuzzy groups. We prove a necessary and sufficient conditions under which the chains of level subgroups of homomorphic images of an a arbitrary fuzzy group can be obtained from that of the fuzzy groups .

Also, we find the chains of level subgroups of homomorphic images and pre-images of arbitrary fuzzy groups.

Key ward:- Fuzzy Groups**,** Group Homomorphism**.**

1.Introduction

If X is a non- empty set then a function $\mu: X \to [0,1]$ is called a fuzzy subset of X [1] . A fuzzy subset μ of G is said to be fuzzy subgroup of G if and only if $\mu(xy) \geq \min\{\mu(x), \mu(y)\} and \mu(x) = \mu(x^{-1})$ [2] .

It is easy to see that if μ is fuzzy subgroup of G, then $\mu(e) \geq \mu(x), \forall x \in G$.[3]

We say that μ has the sup-property if every non- empty subset of Im(μ) has a maximal element.[4], [5] .

If μ is a fuzzy subset of G, then the subset $\mu_t = \{x \in G; \mu(x) \geq t\}$, t$\in$ [0, 1] is called the level subset of μ in G and $\mu_t^* = \{x \in G; \mu(x) > t\}$ is called the strong level subset of μ in G when t = 0 the subset μ_0^* is called support of μ in G and it will be denoted by μ^* [6], [7] .

If λ is a fuzzy subgroup of G, then the level subsets λ_t of λ in G and the strong level subsets λ_t^* of λ in G , t$\in$ $[0, \lambda(e)]$, are subgroups of G and viseversa [8] .

If $t_1, t_2 \in$ Im(μ) such that $t_1 \neq t_2$, then obviously, $\mu_{t_1} \neq \mu_{t_2}$. Further, if Im(μ) = $\{t_i : i = 1,2,..,n\}$ where $t_1 > t_2 > ... > t_n$, then the level subgroups of μ form a chain of subgroups of G. C(μ) $\equiv \mu_{t_1} \subseteq \mu_{t_2} \subseteq ... \subseteq \mu_{t_n} = G$ [2] .

Let $f: G \to H$ be a homomorphism of groups , λ be a fuzzy subgroup of G , μ a fuzzy subgroup of H . Then $f^{-1}(\mu) = \mu(f(x)), \forall x \in G,$

$$f(\lambda)(y) = \begin{cases} \text{Sub}\{\lambda(x); x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \Phi; \\ 0 & \text{if } f^{-1}(y) = \Phi; \quad \forall y \in H \end{cases}$$

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and the fuzzy sets $f(\lambda)$ and $f^{-1}(\mu)$ are fuzzy subgroups of H and G respectively [7], [9]. Now let $f : X \to Y$ be a function and $\lambda[\mu]$ be a fuzzy subset of X[Y]. Then we say that λ is f - invariant if $\lambda(x_1) = \lambda(x_2)$ whenever $f(x_1) = f(x_2), x_1, x_2 \in X$.[5]

2. Homomorphic pre-images of fuzzy groups

In this section, we prove necessary and sufficient conditions under which the chains of level subgroups of homomorphic pre-images of an arbitrary fuzzy group can be obtained from that of the fuzzy group.

Let $f : G \to H$ is a group homomorphism and μ is a fuzzy subgroup of H

We shall denote by $f^{-1}(C(\mu))$ the chain consisting of inverse images under f of members of $C(\mu)$. λ is a fuzzy subgroup of G.

Proposition (2.1)

If μ is a fuzzy subgroup of H and $\{\mu_{t_j} \mid j \in J\}$ is the collection of all level subgroups of μ, then $\{f^{-1}(\mu_{t_j}) \mid j \in J\}$ is the collection of all level subgroups of $f^{-1}(\mu)$.

Proof

Let $\lambda = f^{-1}(\mu)$ and $t \in [0,1]$. Then :

$x \in \lambda_t \Leftrightarrow f^{-1}(\mu) \geq t \Leftrightarrow \mu(f(x)) \geq t \Leftrightarrow f(x) \in \mu_t \Leftrightarrow x \in f^{-1}(\mu_t)$. Hence

$\lambda_t = f^{-1}(\mu_t) \ \forall \ t \in [0, 1]$..........................(1)

In particular, we have : $\lambda_{t_j} = f^{-1}(\mu_{t_j}) \ \forall \ j \in J$.

If λ has a level subgroup λ_t which does not belong to $\{f^{-1}(\mu_{t_j}) \mid j \in J\}$ then μ must have a level subgroup μ_t which does not belong to $\{\mu_{t_j} \mid j \in J\}$ such that (1) holds. This is a contradiction. Hence the result.

We observe from the following example that some of the $f^{-1}(\mu_{t_j})$'s may be equal so that $C(f^{-1}(\mu))$ has fewer components than $C(\mu)$.

Example (2.2)

Let G = {1, -1, i, -i } and H = { e, (12), (13), (23), (123), (132) }.

Then G is a group w. r. t. the usual multiplication of numbers and H is the permutation group of degree three, with e as identity transformation.

Define $f : G \to H$ by $f(x) = e, \forall \ x \in G$. Then f is a group homomorphism. Define $\mu : H \to [0, 1]$ by :

$\mu(e) = 1, \mu((12)) = 0.5, \mu(x) = 0.3, \forall \ x \in H \setminus \{e, (12)\}$.

Then μ is a fuzzy subgroup of H with level subgroups :

$\mu_1 = \{e\}$, $\mu_{0.5} = \{e, (12)\}$, $\mu_{0.3} = H$. But $\lambda = f^{-1}(\mu)$ is defined by :

$\lambda(x) = 1$ for every $x \in G$. Hence, $\lambda_1 = \lambda_{0.5} = \lambda_{0.3} = G$.

Now, we proceed to derive a necessary and sufficient condition for the distinctness of all the $f^{-1}(\mu_{t_j})$. For $t \in Im(\mu)$, we define : $F_\mu(t) = \{ x \in G \mid \mu(x) = t \}$.

Theorem (2.3)

Let $f: G \to H$ be a group homomorphism and μ is a fuzzy subgroup of H with $Im(\mu)=\{t_j \mid j\in J\}$ where J is a countable index set.

Then $f^{-1}(\mu_{t_j})$ are all distinct if and only if : $f(G)\cap F_\mu(t_j) \neq \emptyset, \forall j\in J$.

Proof

Assume that $f^{-1}(\mu_{t_j})$, $j\in J$, are all distinct. Let e* denote the identity element in H. Since f is a homomorphism, $e^*\in f(G)$.
Also, $t_0 \geq t_j$ for every $j\in J$ and hence $\mu(e^*) = t_0$. Hence, $e^*\in F_\mu(t_0)$. Therefore, $f(G)\cap F_\mu(t_j)\neq\emptyset$.
Now, suppose $f(G)\cap F_\mu(t_j)\neq\emptyset$ is empty for some $p > 0$.

Since $t_{p-1} > t_p$, we have $\mu_{t_{p-1}} \in \mu_{t_p}$ and hence $f^{-1}(\mu_{t_{p-1}}) \subseteq f^{-1}(\mu_{t_p})$.

Now, $x\in f^{-1}(\mu_{t_p}) \Rightarrow f(x)\in \mu_p \cup F_\mu(t_j) \Rightarrow f(x)\in \mu_{t_{p-1}}$

since $f(G)\cap F_\mu(t_j) = \emptyset$. $\Rightarrow x\in f^{-1}(\mu_{t_{p-1}})$.

Hence, $f^{-1}(\mu_{t_p}) \subseteq f^{-1}(\mu_{t_{p-1}})$ and therefore : $f^{-1}(\mu_{t_p}) = f^{-1}(\mu_{t_{p-1}})$.

This contradicts the assumption that $f^{-1}(\mu_{t_j})$ are all distinct.
Hence, $f(G)\cap F_\mu(t_j)\neq\emptyset, \forall j\in J$.

Assume that $f^{-1}(\mu_{t_j})$'s are not all distinct. Then we can find $p,q\in J$ such that $t_p\neq t_q$ and $f^{-1}(\mu_{t_p}) = f^{-1}(\mu_{t_q})$……………..(2)
We assume that $t_p < t_q$. Since $f(G)\cap F_\mu(t_p)$ is non-empty, there exists $x\in G$ such that $f(x)\in F_\mu(t_p)$. This implies that $\mu(f(x)) = t_p$.
Since $t_p < t_q$, we have, $f(x)\in \mu_{t_p}$ and $f(x)\notin \mu_{t_q}$. Therefore :

$x\in f^{-1}(\mu_{t_p})$ and $x\notin f^{-1}(\mu_{t_q})$. This contradicts (2). Therefore $f^{-1}(\mu_{t_j})$ are all distinct.

Remark (2.4)
It can be observed from the proof that the second part of the proof in the above theorem hold even when J is uncountable.

If f is a surjection, then $f(G)\cap F_\mu(t_j)\neq\emptyset, \forall j\in J$; and hence $f^{-1}(\mu_{t_j})$ are all distinct.

Corollary (2.5)

If $Im(\mu) = \{t_j \mid j\in J\}$ and $f(G)\cap F_\mu(t_j)\neq\emptyset, \forall j\in J$, then : $C(f^{-1}(\mu)) \equiv f^{-1}(C(\mu))$. In particular, if $J=\{1, 2, \dots, n\}$ and $t_1 > t_2 > \dots > t_n$

then : $C(f^{-1}(\mu)) \equiv f^{-1}(\mu_{t_1}) \subset f^{-1}(\mu_{t_2}) \subset \dots \subset f^{-1}(\mu_{t_n})$.

Proof :
The result follows from theorem (2.3).

3.Homomorphic images of fuzzy groups.

In this section, we study the relationship between $C(\lambda)$ and $C(f(\lambda))$. And prove that if λ is a fuzzy subgroup of G with $Im(\lambda) = \{t_j \mid j=1, 2, \ldots, n\}$ such that $t_1 > t_2 > \ldots > t_n$ and if $f : G \to H$ is a surjective group homomorphism,

then the chain $f(\lambda_{t_1}) \subseteq f(\lambda_{t_2}) \subseteq \ldots \subseteq f(\lambda_{t_n})$ contains all level subgroups of $f(\lambda)$.

In the following proposition, we remove the restriction on the finiteness of $|Im(\lambda)|$.

Proposition (3.1)

If f is a surjection, λ has sup-property and $\{\lambda_{t_j} \mid j \in J\}$ is the collection of all level subgroups of λ, then $\{f(\lambda_{t_j}) \mid j \in J\}$ is the collection of all level subgroups of $f(\lambda)$.

Proof

Let $\mu = f(\lambda)$ and $t \in [0, 1]$.

Then $u \in \mu_t \Rightarrow \mu(u) \geq t \Rightarrow \sup\{\lambda(x) \mid x \in f^{-1}(u)\} \geq t$.

Since λ has sup-property ,this implies that $\lambda(x_0) \geq t$, for some $x_0 \in f^{-1}(u)$. Then $x_0 \in \lambda_t$ and hence $f(x_0) = u \in f(\lambda_t)$.

Therefore, we have $\mu_t \subseteq f(\lambda_t)$.

Now , if $u \in f(\lambda_t)$ then $u = f(x)$ for some $x \in \lambda_t$ and hence.

$\mu(u) = \sup\{\lambda(z) \mid z \in f^{-1}(u)\} = \sup\{\lambda(z) \mid f(z) = f(x)\} \geq \lambda(x) \geq t$

(Since $x \in \lambda_t$). Therefore $u \in \mu_t$ and hence $f(\lambda_t) \subseteq \mu_t$.

Thus we have $\mu_t = f(\lambda_t)$ for every $t \in [0, 1]$............(3)

In particular, $\mu_{t_j} = f(\lambda_{t_j})$, $\forall j \in J$. Hence all $f(\lambda_{t_j})$'s are level subgroups of $\mu = f(\lambda)$ Also, it follows from (3) and the assumption that these are the only level subgroups of μ.

The following example shows that surjectiveness of f, in the above proposition, is essential.

Example (3.2)

Let G = {1, -1} and H = {1, -1, i, -i }.

Define $f : G \to H$ by $f(x) = x$, $\forall x \in G$. Then f is a non-surjective group homomorphism .

Define $\lambda : G \to [0, 1]$ by $\lambda(1) = 0.3$ and $\lambda(-1) = 0.1$.

Then λ is a fuzzy subgroup of G having sup-property. The level subgroups of λ are $\lambda_{0.3} = \{1\}$ and $\lambda_{0.1} = G$. Now, $\mu = f(\lambda)$ is defined by :

$\mu(1) = 0.3$, $\mu(-1) = 0.1$, $\mu(i) = \mu(-i) = 0$. Hence the level subgroups of μ are $\mu_{0.3} = f(\lambda_{0.3}) = \{1\}$, $\mu_{0.1} = f(\lambda_{0.1}) = \{1, -1\}$ and $\mu_0 = H$. Therefore,

$\{f(\lambda_{0.3}), f(\lambda_{0.1})\}$ does not contain all level subgroups of μ. We observe from the following example that surjectiveness of f does not guarantee the distinctness of all $f(\lambda_{t_j})$.

Example (3.3)

Let G = P_3 and H be the subgroup {e, (12)} of P_3, where P_3 denotes the permutation group of degree three.

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Define $f : G \to H$ by :

$$f(x) = \begin{cases} e & \forall x \in \{e, (123), (132)\} \\ (12) & \forall x \in \{(12), (13), (23)\} \end{cases}$$

Then f is a surjective group homomorphism. Define $\lambda : G \to [0, 1]$ by :

$$\lambda(x) = \begin{cases} 0.5 & \text{if } x = (12) \\ 0.2 & \forall x \in G\backslash\{e, (12)\} \\ 0.9 & \text{if } x = e \end{cases}$$

Then λ is a fuzzy subgroup of G having sup-property. The level subgroups of λ are $\lambda_{0.9} = \{e\}$, $\lambda_{0.5} = \{e,(12)\}$, $\lambda_{0.2} = G$. Now, $f(\lambda)$ is given by $f(\lambda)(e) = 0.9$, $f(\lambda)((12)) = 0.5$ and hence $f(\lambda_{0.9}) = \{e\}$, $f(\lambda_{0.5}) = f(\lambda_{0.2}) = H$.

In the following theorem we obtain a necessary and sufficient condition for the distinctness of all $f(\lambda_{t_j})$.

Theorem (3.4)

If $f : G \to H$ is a surjective group homomorphism and λ is a fuzzy subgroup of G having sup-property and $Im(\lambda)=\{t_j \mid j \in J\}$ where J is a countable index set. Then $\{f(\lambda_{t_j}) , j \in J\}$, are all distinct if and only if λ is f-invariant.

Proof

Suppose $f(\lambda_{t_j})$'s are all distinct . Since $t_j > t_{j+1}$ $\forall j \in J$ we have $\lambda_{t_j} \subset \lambda_{t_{j+1}}$, and hence, $f(\lambda_{t_j}) \subset f(\lambda_{t_{j+1}})$.

Let $x, y \in G$ such that $f(x) = f(y)$. Let $f(\lambda_{t_p})$ be the smallest $f(\lambda_{t_j})$ which contains $f(x)$. If p = 0 .

Then $f(x) = f(y) \in f(\lambda_{t_0})$ and hence

$\lambda(x) = \lambda(y) = \lambda(e)$. If $p \neq 0$. Then $f(x), f(y) \in f(\lambda_{t_p})$ and

$f(x), f(y) \notin f(\lambda_{t_{p-1}})$. Hence $x, y \in \lambda_{t_p}$ and $x , y \notin \lambda_{t_{p-1}}$.

Therefore $\lambda(x) = \lambda(y) = t_p$.

Thus, in both cases, we have , $\lambda(x) = \lambda(y)$, and hence λ is f-invariant.

Conversely, Assume that λ is f-invariant. Then for any $z \in H$,

$f(\lambda)(z) = \lambda(x)$, $\forall x \in f^{-1}(z)$……………….(4)

If $f(\lambda_{t_j})$'s are not distinct then there exists $t_p, t_q \in Im(\lambda)$ such that $t_p \neq t_q$ and $f(\lambda_{t_p}) = f(\lambda_{t_q})$. Since $t_p, t_q \in Im(\lambda)$, there exist $x, y \in G$ such that $\lambda(x) = t_p$, and $\lambda(y) = t_q$. Hence by (4) , we have :

$f(\lambda)(f(x)) = t_p$ and $f(\lambda)(f(y)) = t_q$.

Therefore $t_p, t_q \in Im(f(\lambda))$ and hence it follows that : $f(\lambda_{t_p}) \neq f(\lambda_{t_q})$.

This is a contradiction. Hence $f(\lambda_{t_j})$, $j \in J$, are all distinct. □

We observe that the proof of the second part does not require the countability of J. Hence we have following result.

Corollary (3.5)

If $f: G \rightarrow H$ is a surjective group homomorphism and λ is an f-invariant fuzzy subgroup of G having sup-property then :

$C(f(\lambda)) \equiv f(C(\lambda))$.

Proof

The result follows from theorem (3.4) .

Corollary (3.6)

Let $f: G \rightarrow H$ is a surjective group homomorphism and λ be a fuzzy subgroup of G with $Im(\lambda) = \{t_i \mid i=1, 2, \ldots, n\}$ where $t_1 > t_2 > \ldots > t_n$. Then :

(i) $\{f(\lambda_{t_i}) \mid i=1, 2, \ldots, n\}$ contains all level subgroups of $f(\lambda)$.

(ii) $\{f(\lambda_{t_i}), i=1,2,\ldots,n\}$ are all distinct if and only if λ is f-invariant.

(iii) If λ is f-invariant then $Im(f(\lambda)) = Im(\lambda)$ and

$C(f(\lambda)) \equiv f(\lambda_{t_1}) \subseteq f(\lambda_{t_2}) \subseteq \ldots \subseteq f(\lambda_{t_n})$.

Proof

It is straight forward.

Remark (3.7) □

Theorems (2.3) and (3.4) give us methods to obtain the chains of level subgroups of homomorphic images and pre-images of an arbitrary fuzzy group from that of the given fuzzy group .

More specifically if $f(G) \cap F_{\mu}(t) \neq \emptyset$ for every $t \in Im(\mu)$,

then $C(f^{-1}(\mu)) \equiv f^{-1}(C(\mu))$. Further, if f is a surjection and λ is f-invariant, then $C(f(\lambda)) \equiv f(C(\lambda))$.

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حول الزمر الضبابية و زمر التشاكل

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الخلاصة

يهتم هذا البحث بدراسة تأثير تشاكل الزمر في سلاسل الزمر الجزئية المستوية من الزمر الضبابية وأثبتنا الشروط الضرورية و اللازمة للحصول على سلاسل الزمر الجزئية المستوية لصور التشاكل (الصور العكسية) لأي زمرة ضبابية اختيارية بالوقت نفسه تمكنا بوساطة تلك النظريات من إيجاد سلاسل الزمر الجزئية المستوية لصورة التشاكل والصورة العكسية لها في أي زمرة ضبابية اختيارية .

الكلمات المفتاحية :-الزمر الضبابية ، تشاكل الزمر