Ibn Al-Haitham Journal for Pure and Applied Science
مجلة إبن الهيثم للعلوم الصرفة و التطبيقية
No. 2 Vol. 25 Year 2012
2012 السنة 25 المجلد 2 العدد

The Commutator of Two Fuzzy Subsets

L. N. M. Tawfiq , R. H. Shihab
Department of Mathematics, College of Education Ibn Al-Haitham, University of Baghdad.

Received in:2 March 2011 Accepted in: 12 April 2011

Abstract

In this paper we introduce the idea of the commutator of two fuzzy subsets of a group and study the concept of the commutator of two fuzzy subsets of a group .We introduce and study some of its properties .

Key ward: fuzzy set , fuzzy group, normal fuzzy subgroup.

1.Introduction

Applying the concept of fuzzy sets of Zadeh to the group theory, Rosenfeld introduced the notion of a fuzzy group as early as 1971.

The technique of generating a fuzzy group (the smallest fuzzy group) containing an arbitrarily chosen fuzzy set was developed only in 1992 by Malik , Mordeson and Nair, [1] .

In this paper, we use our notion of commutator of two fuzzy subsets of a group.

Now we introduce the following definitions which is necessary and needed in the next section :

Definition 1.1 [1], [2]

A mapping from a nonempty set X to the interval [0, 1] is called a fuzzy subset of X .

Next, we shall give some definitions and concepts related to fuzzy subsets of G.

Definition 1.2

Let μ, ν be fuzzy subsets of G, if $\mu(x) \le \nu(x)$ for every $x \in G$, then we say that μ is contained in ν (or ν contains μ) and we write $\mu \subseteq \nu$ (or $\nu \supseteq \mu$).

If $\mu \subseteq \nu$ and $\mu \ne \nu$, then μ is said to be properly contained in ν (or ν properly contains μ) and we write $\mu \subset \nu$ (or $\nu \supset \mu$).[3]

Note that: $\mu = \nu$ if and only if $\mu(x) = \nu(x)$ for all $x \in G$.[4]

Definition 1.3 [3]

Let μ, ν be two fuzzy subsets of G. Then $\mu \cup \nu$ and $\mu \cap \nu$ are fuzzy subsets as follows:

(i) $(\mu \cup \nu)(x) = \max\{\mu(x), \nu(x)\}$

(i) $(\mu \cap v)(x) = \min\{\mu(x), v(x)\}$, for all $x \in G$

Then $\mu \cup v$ and $\mu \cap v$ are called the union and intersection of μ and v, respectively.

Definition 1.4[5]

For μ, v are two fuzzy subsets of G, we define the operation $\mu \circ v$ as follows:

$(\mu \circ v)(x) = sup\{min\{\mu(a), v(b)\} \mid a,b \in G \ and \ x = a*b\}$ For all $x \in G$.

We call $\mu \circ v$ the product of μ and v.

Now, we are ready to give the definition of a fuzzy subgroup of a group.

Definition 1.5[1], [6]

A fuzzy subset μ of a group G is a fuzzy subgroup of G if:

(i) $min\{\mu(a), \mu(b)\} \le \mu(a*b)$

(ii) $\mu(a^{-1}) = \mu(a)$, for all $a, b \in G$.

Theorem 1.6 [3]

If μ is a fuzzy subset of G, then μ is a fuzzy subgroup of G, if and only if, μ satisfies the following conditions:

(i) $\mu \circ \mu \subseteq \mu$

(ii) $\mu^{-1} = \mu$

where $\mu^{-1}(x) = \mu(x)$, $\forall x \in G$.

Proposition 1.7 [6]

Let μ be a fuzzy group. Then $\mu(a) \le \mu(e)$ $\forall a \in G$.

Definition 1.8 [7]

If μ is a fuzzy subgroup of G, then μ is said to be abelian if $\forall x, y \in G$, $\mu(x) > 0, \mu(y) > 0$, then $\mu(xy) = \mu(yx)$.

Definition 1.9 [8] , [9]

A fuzzy subgroup μ of G is said to be normal fuzzy subgroup if

$\mu(x*y) = \mu(y*x)$, $\forall x, y \in G$.

2. The Commutator of Two Fuzzy Subsets of a Group

In this section we introduce the idea of the commutator of two fuzzy subsets of a group and prove some of its properties.

Definition 2.1

Let λ and μ be two fuzzy subsets of G. The commutator of λ and μ is the fuzzy subgroup $[\lambda, \mu]$ of G generated by the fuzzy subset (λ, μ) of G which is defined as follows for any $x \in G$:

Ibn Al-Haitham Journal for Pure and Applied Science | مجلة إبن الهيثم للعلوم الصرفة و التطبيقية
No. 2 | Vol. 25 | Year 2012 | 2012 السنة | 25 المجلد | 2 العدد

$$(\lambda,\mu)(x)=\begin{cases}\sup\{\lambda(a)\wedge\mu(b)\} & \text{if } x \text{ is a commutator } x=[a,b] \\ 0 & \text{otherwise,}\end{cases}$$

Next, we will introduce some theorems about the commutator of two fuzzy subsets of a group which is useful in fuzzy mathematics .

Theorem 2. 2

If A, B are subsets of G, then $[\chi_A,\chi_B]=\chi_{[A,B]}$, where for all $x\in G$:

$$\chi_A(a)=\begin{cases}1, & \text{if } a\in A\\ 0, & \text{if } a\notin A\end{cases}$$

Proof:

and $[\chi_A,\chi_B]=\langle(\chi_A,\chi_B)\rangle$

$$(\chi_A,\chi_B)=\begin{cases}\sup\limits_{x=[a,b]}\{\chi_A(a)\wedge\chi_B(b)\}, & \text{if } x \text{ is a commutator}\\ 0, & \text{otherwise}\end{cases}$$

Then:

$$[\chi_A,\chi_B]=\begin{cases}1 & \text{if } x=[a,b]\in[A,B]\\ 0 & \\ 0 & \text{otherwise}\end{cases} \qquad (1)$$

On the other hand ,

$$\begin{cases}1 & \text{if } x\in[A,B]\\ 0 & \\ 0 & \text{otherwise}\end{cases} \qquad (2)$$

Ibn Al-Haitham Journal for Pure and Applied Science | مجلة إبن الهيثم للعلوم الصرفة و التطبيقية
No. 2 | Vol. 25 | Year 2012 | 2012 السنة | 25 المجلد | 2 العدد

$\chi_{[A,B]}(x) =$

From (1) and (2), we get $[\chi_A, \chi_B] = \chi_{[A,B]}$.

Theorem 2. 3

For any two fuzzy subsets λ, μ of G, $[\lambda, \mu] = [\mu, \lambda]$

Proof :

The result follows from definition (2.1) and definition(1.5).

For more explanation we give the following example:

Example 2. 4

Let λ and μ be two fuzzy subsets of S_3 (the group of all permutations on the set $\{1,2,3,\}$) defined as follows, for any x $\in$ S_3 :

$$\lambda(x) = \begin{cases} 1 & \text{if } x = \{e\} \\ ½ & \text{if } x \in A_3 - \{e\} \\ ¼ & \text{if } x \in S_3 - A_3 \end{cases}$$

$$\mu(x) = \begin{cases} 1 & \text{if } x = \{e\} \\ ⅓ & \text{if } x \in \{(12),(13),(23)\} \\ 0 & \text{otherwise} \end{cases}$$

By definition (2.1) :

$$(\lambda, \mu)(x) = \begin{cases} \sup\{\lambda(a) \wedge \mu(b)\} \quad x = [a,b] & \text{if } x \text{ is a commutator} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } x = \{e\} \\ \frac{1}{3} & \text{if } x \in A_3 - \{e\} \\ 0 & \text{otherwise} \end{cases}$$

Hence $[\lambda, \mu] = \langle(\lambda, \mu)\rangle = \begin{cases} 1 & \text{if } x = \{e\} \\ \frac{1}{3} & \text{if } x \in A_3 - \{e\} \\ 0 & \text{otherwise} \end{cases}$

On the other hand :

$$(\mu, \lambda)(x) = \begin{cases} 1 & \text{if } x = \{e\} \\ \frac{1}{3} & \text{if } x \in A_3 - \{e\} \\ 0 & \text{otherwise} \end{cases}$$

Hence $[\mu, \lambda] = \langle(\mu, \lambda)\rangle = \begin{cases} 1 & \text{if } x = \{e\} \\ \frac{1}{3} & \text{if } x \in A_3 - \{e\} \\ 0 & \text{otherwise} \end{cases}$

Thus $[\lambda, \mu] = [\mu, \lambda]$.

Theorem 2. 5

If λ, μ, β and δ are fuzzy subsets of G such that $\lambda \subseteq \mu$ and $\beta \subseteq \delta$, then: $[\lambda, \beta] \subseteq [\mu, \delta]$.

Proof:

$[\lambda, \beta] = \langle(\lambda, \beta)\rangle$, by definition (2.1)

for all $x \in G$,

$$(\lambda, \beta)(x) = \begin{cases} \sup\{\lambda(a) \wedge \beta(b)\} & \text{if } x \text{ is a commutator} \\ x = [a, b] & \end{cases}$$

$$\begin{cases} \sup\{\mu(a) \wedge \delta(b)\} & \text{if } x \text{ is a commutator} \\ x = [a, b] & \end{cases}$$

£

$= (\mu, \delta)(x)$

Thus,

$$[\lambda, \beta] = \langle(\lambda, \beta)\rangle \leq \langle(\mu, \delta)\rangle = [\mu, \delta]$$

Hence, $[\lambda, \beta] \subseteq [\mu, \delta]$. .

Corollary 2. 6

If λ, μ are fuzzy subsets of G such that $\lambda \subseteq \mu$, then $[\lambda, \delta] \subseteq [\mu, \delta]$ for every fuzzy subset δ of G.

Proof:

The result follows from theorem (2. 5) by taking $\beta = \delta$.

Now, we introduce an important concept about the fuzzy subset.

Definition 2. 7

Let λ be a fuzzy subset of G. Then the tip of λ is the supremum of the set $\{\lambda(x) | x \in G\}$.

Theorem 2. 8

Let λ *and* μ be fuzzy subsets of G. Then the tip of $[\lambda, \mu]$ is the minimum of tip of λ and tip of μ.

Proof:

We want to prove that the tip of $[\lambda, \mu] = tip\ of\ \lambda \wedge tip\ of\ \mu$

Let tip of $\lambda = \sup\{\lambda(x) / x \in G\} = L$

And, let tip of $\mu = \sup\{\mu(x) / x \in G\} = M$

Such that $L, M \in [0,1]$

Now,

Tip of (λ, μ)

$$\begin{aligned}
&= \sup\{(\lambda, \mu)(x) / x \in G\} \\
&= \sup\{\sup\{\lambda(a) \wedge \mu(b)\} / x = [a,b], x \in G\} \\
&= \sup\sup\{\lambda(a) \wedge \mu(b) / x = [a,b], x \in G\} \\
&= \sup\{\lambda(a) \wedge \mu(b) / x = [a,b]\ and\ a,b \in G\ \} \\
&= \{\sup \lambda(a) / a \in G\} \wedge \{\sup \mu(b) / b \in G\} \\
&= L \wedge M
\end{aligned}$$

Since, $[\lambda, \mu] = \langle \lambda, \mu \rangle$

Therefore, tip of $[\lambda, \mu] = L \wedge M$

That is tip of $[\lambda, \mu] = tip\ of\ \lambda \wedge tip\ of\ \mu$.

Theorem 2. 9

Let λ, μ be fuzzy subsets of G. If $S(\lambda) = H$ and $S(\mu) = K$, then : $S([\lambda, \mu]) = [H, K]$.

Proof:

First, we have $S(\lambda) = \{x \in G \mid \lambda(x) > 0\} = H$ and $S(\mu) = \{x \in G \mid \mu(x) > 0\} = K$. Then :

$[H, K]$

$$= \{\langle [a,b] \rangle / a \in H, b \in K\}$$

$$= \{\langle [a,b] \rangle / \lambda(a) > 0\ \ \mu(b) > 0\}$$

$$= \{x / x \text{ is a commutaltor }\} \quad \_\_\_\_\_\_\_\_ \quad (1)$$

and,

$$S([\lambda, \mu]) = S(\langle \lambda, \mu \rangle)$$

$$= S\left\langle \begin{cases} \sup\{\lambda(a) \wedge \mu(b)\} & \text{if } x \text{ is a commutator} \\ x = [a,b] & \\ 0 & \text{otherwise} \end{cases} \right\rangle$$

$$= \{x / x : \text{ is a commutator }\} \quad \_\_\_\_\_\_\_\_ \quad (2)$$

From (1) and (2), we get $S([\lambda, \mu]) = [H, K]$.

The following example illustrates theorems (2. 8) and (2. 9).

Example 2. 10

Let λ and μ be a fuzzy subsets of S_3 which are defined as follows:

$$\lambda(x) = \begin{cases} 1 & \text{if } x = \{e\} \\ 1/3 & \text{if } x \in A_3 - \{e\} \\ 1/4 & \text{if } x \in S_3 - A_3 \end{cases}$$

Ibn Al-Haitham Journal for Pure and Applied Science | مجلة إبن الهيثم للعلوم الصرفة و التطبيقية

No.	2	Vol.	25	Year	2012	2012	السنة	25	المجلد	2	العدد

and,

$$\mu(x) = \begin{cases} ½ & \text{if } x \in A_3 \\ 0 & \text{otherwise} \end{cases} \quad , x \in S_3$$

Then, tip of $\lambda = 1$ and tip of $\mu = ½$

$$(\lambda, \mu)(x) = \begin{cases} \sup\{\lambda(a) \wedge \mu(b)\} & \text{if } x \text{ is a commutator } x = [a,b] \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} ½ & \text{if } x = \{e\} \\ ⅓ & \text{if } x \in A_3 - \{e\} \\ 0 & \text{otherwise} \end{cases} \quad , x \in S_3$$

Since, $[\lambda, \mu] = \langle (\lambda, \mu) \rangle$. Then :

$$[\lambda, \mu](x) = \begin{cases} ½ & \text{if } x = \{e\} \\ ⅓ & \text{if } x \in A_3 - \{e\} \\ 0 & \text{otherwise} \end{cases}$$

Then tip of $[\lambda, \mu] = ½$

$$= tip\ of\ \lambda \wedge tip\ of\ \mu$$

Also,

$S(\lambda)=S_3$ and $S(\mu)=A_3$

$$[S(\lambda),S(\mu)]=\{\langle[a,b]\rangle \mid a\in S(\lambda), b\in S(\mu)\}=A_3$$

Also, $S([\lambda,\mu])=A_3$

Then, we get : $S([\lambda,\mu])=[S(\lambda),S(\mu)]$.

Next, we will give and prove the following propositions, which we will be needed later.

Proposition 2. 11

If λ is a fuzzy subgroup of G, then : $[\lambda,\lambda]\subseteq \lambda$.

Proof:

For all $x\in G$,

$$(\lambda,\lambda)(x)=\begin{cases}\sup\{\lambda(a)\wedge\lambda(b)\} & \text{if } x \text{ is a commutator} \\ x=[a,b] & \\ 0 & \text{otherwise}\end{cases}$$

$$=\begin{cases}\sup\{\lambda(a)\wedge\lambda(b)\wedge\lambda(a^{-1})\wedge\lambda(b^{-1})\} & \text{if } x \text{ is a commutator} \\ x=[a,b] & \\ 0 & \text{otherwise}\end{cases}$$

$$\leq\begin{cases}\sup\{\lambda(aba^{-1}b^{-1})\} & \text{if } x \text{ is a commutator} \\ x=[a,b] & \\ 0 & \text{otherwise}\end{cases}$$

$$=\lambda(x)$$

Ibn Al-Haitham Journal for Pure and Applied Science | مجلة إبن الهيثم للعلوم الصرفة و التطبيقية
No. 2 Vol. 25 Year 2012 | 2012 السنة 25 المجلد 2 العدد

That is $(\lambda,\lambda) \subseteq \lambda$.

Hence $[\lambda,\lambda] \subseteq \lambda$.

From theorem (2. 5) and proposition (2. 11) we obtain the following corollary :

Corollary 2. 12

Let $\lambda, \mu, \beta, \delta, \nu$ and α be fuzzy subsets of G. If $[\beta,\delta] \subseteq [\lambda,\mu]$ and $[\nu,\alpha] \subseteq [\lambda,\mu]$. Then $[[\beta,\delta],[\nu,\alpha]] \subseteq [\lambda,\mu]$.

Proof:

Since $[\beta,\delta] \subseteq [\lambda,\mu]$ and $[\nu,\alpha] \subseteq [\lambda,\mu]$

Then : $[[\beta,\delta],[\nu,\alpha]] \subseteq [[\lambda,\mu],[\lambda,\mu]] \subseteq [\lambda,\mu]$

Hence : $[[\beta,\delta],[\nu,\alpha]] \subseteq [\lambda,\mu]$.

Proposition 2. 13

Let λ, μ be fuzzy subsets of G. Then: $[\lambda,\mu] \circ [\lambda,\mu] = [\lambda,\mu]$.

Proof:

From definition (2. 1), $[\lambda,\mu]$ is fuzzy subgroup of G and by theorem (1.6) :

$[\lambda,\mu] \circ [\lambda,\mu] \subseteq [\lambda,\mu]$ (1)

Now, let $x \in G$

$$\begin{aligned}([\lambda,\mu] \circ [\lambda,\mu])(x) &= \sup\{[\lambda,\mu](a) \wedge [\lambda,\mu](b), x = a*b\}\\ &\geq \{[\lambda,\mu](x) \wedge [\lambda,\mu](e), x = xe\}\\ &= [\lambda,\mu](x)\end{aligned}$$

That is $[\lambda,\mu] \subseteq [\lambda,\mu] \circ [\lambda,\mu]$ (2)

From (1) and (2), we get : $[\lambda,\mu] \circ [\lambda,\mu] = [\lambda,\mu]$.

Proposition 2. 14

Let $\lambda, \mu, \beta, \delta, \nu$ and α be fuzzy subsets of G, such that $[\beta,\delta] \subseteq [\lambda,\mu]$ and $[\nu,\alpha] \subseteq [\lambda,\mu]$. Then $[\beta,\delta] \circ [\nu,\alpha] \subseteq [\lambda,\mu]$.

Proof:

For all $x \in G$,

$$\begin{aligned}([\beta,\delta] \circ [\nu,\alpha])(x) &= \sup\{[\beta,\delta](a) \wedge [\nu,\alpha](b), x = a*b\}\\ &\leq \sup\{[\lambda,\mu](a) \wedge [\lambda,\mu](b), x = a*b\}\\ &= ([\lambda,\mu] \circ [\lambda,\mu])(x)\\ &= [\lambda,\mu](x) \quad \text{(by proposition (2. 13))}\end{aligned}$$

Hence, $[\beta,\delta] \circ [\nu,\alpha] \subseteq [\lambda,\mu]$.

Now, we can give the following corollary:

Corollary 2. 15

If $\lambda, \mu, \beta, \delta, \nu$ and α are fuzzy subsets of G, such that $[\lambda,\beta] \subseteq [\mu,\delta]$, then :

(i) $[\lambda,\beta] \circ [\nu,\alpha] \subseteq [\mu,\delta] \circ [\nu,\alpha]$

(ii) $[\lambda,\beta] \circ [\mu,\delta] \subseteq [\mu,\delta]$

(iii) If $[\lambda,\beta] \subseteq [\nu,\alpha]$, then $[\lambda,\beta] \subseteq [\mu,\delta] \circ [\nu,\alpha]$.

Proof:

The result follows from proposition (2. 14).

Ibn Al-Haitham Journal for Pure and Applied Science
مجلة إبن الهيثم للعلوم الصرفة و التطبيقية
No. 2 Vol. 25 Year 2012
2012 السنة 25 المجلد 2 العدد

Next, we will give and prove the following proposition :

Proposition 2. 16

Let $[l, b], [m, d]$ be two fuzzy subgroups of G. Then $[l, b](e) = [m, d](e)$ if and only if, $[l, b] \subseteq [l, b] \circ [m, d]$ and $[m, d] \subseteq [l, b] \circ [m, d]$.

Proof:

First, if $[l, b](e) = [m, d](e)$, we prove :

$[l, b] \subseteq [l, b] \circ [m, d]$ and $[m, d] \subseteq [l, b] \circ [m, d]$.

Let $x \in G$,

$$\begin{aligned}([l, b] \circ [m, d])(x) &= \sup\{[l, b](a) \wedge [m, d](b), x = a * b\} \\ &\geq \{[l, b](x) \wedge [m, d](e), x = x * e\} \\ &= \{[l, b](x) \wedge [l, b](e), x = x * e\} \\ &= [l, b](x)\end{aligned}$$

That is, $[l, b](x) \leq ([l, b] \circ [m, d])(x)$ for all $x \in G$.

Hence, $[l, b] \subseteq [l, b] \circ [m, d]$

Also,

$$\begin{aligned}([l, b] \circ [m, d])(x) &= \sup\{[l, b](a) \wedge [m, d](b), x = a * b\} \\ &\geq \{[l, b](e) \wedge [m, d](x), x = e * x\} \\ &= \{[m, d](e) \wedge [m, d](x), x = e * x\} \\ &= [m, d](x)\end{aligned}$$

That is $[m, d](x) \leq ([l, b] \circ [m, d])(x)$ *for all* $x \in G$

Hence $[m, d] \subseteq [l, b] \circ [m, d]$.

Conversely, we prove $[l, b](e) = [m, d](e)$.

Suppose $[l, b](e) \neq [m, d](e)$, then if $[l, b](e) \geq [m, d](e)$.

$$\begin{aligned}[l, b](e) &\leq ([l, b] \circ [m, d])(e) \\ &= \sup\{[l, b](a) \wedge [m, d](b), e = a * b\} \\ &\leq \{[l, b](e) \wedge [m, d](b), e = e * e\} \\ &= [m, d](e)\end{aligned}$$

That is $[l, b](e) \leq [m, d](e)$, which is a contradiction .

Now, if $[m, d](e) \geq [l, b](e)$,then in the same way we get a contradiction.

Therefore $[l, b](e) = [m, d](e)$.

References

1 .Malik . D.S. , Mordeson . J. N. and Nair. P. S. ,(1992) " Fuzzy Generators and Fuzzy Direct Sums of Abelian Groups", Fuzzy sets and systems,.50, 193-199,

2.Majeed.S.N., (1999)"On fuzzy subgroups of abelian groups",M.Sc. Thesis, University of Baghdad,.

3.Mordesn J.N.,(1996)"L-subspaces and L-subfields" , .

4.Hussein. R. W., (1999)"Some results of fuzzy rings" , M.Sc. Thesis , University of Baghdad ,.

5.Liu. W.J., (1982) "Fuzzy invariant subgroups and fuzzy ideals", fuzzy sets and systems . l.8. 133-139,.
6.Abou-Zaid. , (1988)" On normal fuzzy subgroups " , J.Facu.Edu.,.13, .
7. Seselja .B and Tepavcevic A. , (1997) "Anote on fuzzy groups" , J.Yugoslav. Oper. Rese ,.7, No.1, pp.49-54, .
8.Gupta K.c and Sarma B.K., (1999) "nilpotent fuzzy groups" ,fuzzy set and systems ,.101,.167-176 ,.
9.Seselja . B . and Tepavcevic A. ,(1996) " Fuzzy groups and collections of subgroups " , fuzzy sets and systems ,.83 ,.85-91 ,.
10.Bandler. W. and Kohout. L. , (2000) Semantics of implication operators and fuzzy relational products, Internat. J. Man- Machine studies 12 89 -116 .

المبادل لمجموعتان جزئيتان ضبابيتان

لمى ناجي محمد توفيق ,رجاء حامد شهاب

قسم الرياضيات – كلية التربية أبن الهيثم – جامعة بغداد.

استلم البحث:ÑĢÇ2 2011 قبل البحث في 12 نيسان 2011.

الخلاصة

يتضمن البحث تقديم فكرة المبادل لمجموعتان جزئيتان ضبابيتان ودراسة مفهوم المبادل لمجموعتان جزئيتان ضبابيتان من زمرة و دراسة خواصها وتقديم البراهين المهمة حول المفهوم .

الكلمات المفتاحية : المجموعات الضبابية ، الزمر الضبابية