Ibn Al-Haitham Journal for Pure and Applied Science | مجلة إبن الهيثم للعلوم الصرفة و التطبيقية

No. 2 Vol. 25 Year 2012 | العدد 2 المجلد 25 السنة 2012

Estimate The Mean of Normal Distribution Via Preliminary Test Shrinkage Technique

S. T.Abdulrahman
Department of Mathematics-Ibn-Al-Haitham College of Education – University of Baghdad.
Received in: 18 December 2011 Accepted in: 11January 2012

Abstract

This paper is concerned with preliminary test single stage shrinkage estimators for the mean (θ) of normal distribution with known variance σ^2 when a prior estimate (θ_0) of the actule value (θ) is available, using specifying shrinkage weight factor $\psi(\cdot)$ as well as pre-test region (R).

Expressions for the Bias, Mean Squared Error [MSE($\cdot$)] and Relative Efficiency [R.Eff.($\cdot$)] of proposed estimators are derived. Numerical results and conclusions are drawn about selection different constants including in these expressions. Comparisons between suggested estimators with respect to usual estimators in the sense of Relative Efficiency are given. Furthermore, comparisons with the earlier existing works are drawn to shown the usefulness of the proposed estimators.

Key Words: Prior Estimate, Shrinkage Estimator, Shrinkage weight factor, Pre-test Region, Bias Ratio, Mean Squared Error and Relative Efficiency.

1.Introduction

Some time we may have a prior estimate value (point guess) of the parameter to be estimated. If this value is in the vicinity of the true value, the shrinkage technique is useful to get an improved estimator. Thompson in [1], Mehta and Srinivasan in [2], Singh at el in [3] and others suggested shrunken estimators for different distributions when a prior estimate or guess point is available. They showed that these estimators perform better in the term of Mean Square Error when a guess value θ_0 close to the true value θ.

Assume that $x_1, x_2, \ldots, x_n$ be a random sample of size (n) from a normal population with known variance (σ^2) and un-known mean (θ).

In conventional notation, we write $x \sim N(\theta,\sigma^2)$ …(1)

Preliminary test estimator in Thompson [1] is considered for estimating the parameter θ in previous model of distribution in (1) when a guess point (prior estimate) θ_0 is available about θ due the past knowledge or similar cases.

From the empirical studies it has been established that the shrinkage estimators performs better than the usual estimator when the guess point be very close to the true value of the parameter. Therefore to make sure whether θ is closed to θ_0 or not, we may test $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$, so we denote by R to the critical region for above test.

Thompson suggested shrinking the usual estimator $\hat{\theta}$ of θ towards the prior guess point θ_0 and proposed the estimator $\tilde{\theta} = \psi(\hat{\theta})\hat{\theta} + (1-\psi(\hat{\theta}))\theta_0$, where $(1-\psi(\hat{\theta}))$ represents the experimenters belief in the guess point θ_0. He was found the estimator $\tilde{\theta}$ is more efficient than $\hat{\theta}$ if the true value θ is close to θ_0 (H_0 accepted) but may be less efficient otherwise, therefore to resolve the uncertainty that a guess point value is approximately the true value or

Ibn Al-Haitham Journal for Pure and Applied Science
No. 2 Vol. 25 Year 2012
مجلة إبن الهيثم للعلوم الصرفة و التطبيقية
العدد 2 المجلد 25 السنة 2012

not, a preliminary test of significance may be employed. So he take the usual estimator $\hat{\theta}$ when θ is far a way from θ_0 (H_0 rejected) after he made the preliminary test.

Thus, the preliminary test shrunken estimator has the following form

$$\tilde{\theta} = \begin{cases} \psi(\hat{\theta})\hat{\theta} + (1-\psi(\hat{\theta}))\theta_0 & , \text{if } \hat{\theta} \in R \\ \hat{\theta} & , \text{if } \hat{\theta} \notin R \end{cases} \quad \ldots(2)$$

where R is the preliminary test region for acceptance the null hypothesis H_0 as we mentioned above, $\hat{\theta}$ is the usual estimator of θ, $\psi(\hat{\theta})$ is a shrinkage weight factor such that $0 \le \psi(\hat{\theta}) \le 1$ which may be a function of $\hat{\theta}$ or may be a constant (ad hoc basis).
Several authors had been studied a preliminary test shrinkage estimator which is defined in (2) for special population by choosing different weight factors $\psi(\hat{\theta})$. See for example [4], [5], [6], [7], [8], [9], [10], [11],[12]and [13].

The **aim** of this paper is to modify the preliminary test shrunken estimator which is defined in (2) for estimate the parameters (θ) of the proposed distribution model (1).
Therefore, the form of the proposed modify preliminary test shrinkage estimator is as below:-

$$\tilde{\theta}_{PT} = \begin{cases} \psi_1(\hat{\theta})\hat{\theta} + (1-\psi_1(\hat{\theta}))\theta_0 & , \text{if } \hat{\theta} \in R \\ \psi_2(\hat{\theta})\hat{\theta} + (1-\psi_2(\hat{\theta}))\theta_0 & , \text{if } \hat{\theta} \notin R \end{cases} \quad \ldots(3)$$

where $\psi_i(\hat{\theta})$, (i = 1,2) is a shrinkage weight factor such that $0 \le \psi_i(\hat{\theta}) \le 1$.

The expressions for Bias, Mean Square Error and Relative Efficiency of the estimator $\tilde{\theta}_{PT}$ above are derived. Numerical results of these expressions were made to show the validity and the usefulness of the proposed estimator when it compares with the usual and existing estimators.

2. Preliminary Test Single Stage Shrunken Estimator $\tilde{\theta}_{PT1}$

In this section, we want to estimate the parameter θ using the following preliminary test Shrunken estimator:

$$\tilde{\theta}_{PT1} = \begin{cases} \psi_1(\hat{\theta})\hat{\theta} + (1-\psi_1(\hat{\theta}))\theta_0 & , \text{if } \hat{\theta} \in R \\ \psi_2(\hat{\theta})\hat{\theta} + (1-\psi_2(\hat{\theta}))\theta_0 & , \text{if } \hat{\theta} \notin R \end{cases} \quad \ldots(4)$$

where $\psi_i(\hat{\theta})$, (i = 1,2) is shrinkage weight factor such that $0 \le \psi_i(\hat{\theta}) \le 1$ and $\hat{\theta}$ is usual estimator of θ as well as R is the pretest region for acceptance of testing the hypothesis $H_0{:}\theta = \theta_0$ vs. the hypothesis $H_1{:}\theta \ne \theta_0$ with level of significance α using test statistic

$$T(\hat{\theta}/\theta_0) = \frac{\hat{\theta} - \theta_0}{s/\sqrt{n}} .$$

$$\text{i.e. } R = [-\theta_0 - Z_{\alpha/2}\sqrt{\frac{s^2}{n}}, -\theta_0 + Z_{\alpha/2}\sqrt{\frac{s^2}{n}}] \quad \ldots(5)$$

where $Z_{\alpha/2}$ is the 100(α/2) percentile point of the standard normal distribution.

In the estimator $\tilde{\theta}_{PT1}$ which is defined in (4), we assume that $\psi_1(\theta) = 0$ and $\psi_2(\theta) = e^{-10/n}$.

The expressions for Bias and Mean Square Error (MSE) of $\tilde{\theta}_{PT1}$ are respectively given as below:-

$$\text{Bias}(\tilde{\theta}_{PT1}|\theta,R)=E(\tilde{\theta}_{PT1})-\theta$$
$$=\int_{R}-(\theta-\theta_0)]f(\hat{\theta}|\theta)d\hat{\theta}+\int_{\bar{R}}[e^{-10/n}(\hat{\theta}-\theta_0)+(\theta_0-\theta)]\,f(\hat{\theta}|\theta)d\hat{\theta}$$

where $\bar{R}$ is the complement region of R in real space, $\hat{\theta}\sim N(\mu,\frac{\sigma^2}{n})$ and

$$f(\hat{\theta}|\theta,\sigma^2)=\frac{\sqrt{n}}{\sigma\sqrt{2\pi}}\exp[-n(\hat{\theta}-\theta)^2/2\sigma^2],\ -\infty<\hat{\theta}<\infty,-\infty<\theta<\infty,\sigma^2\geq 0.$$

The previous expression will result

$$\text{Bias}(\tilde{\theta}_{PT1}|\theta,R)=\frac{\sigma}{\sqrt{n}}\{-\lambda[1-e^{-10/n}+e^{-10/n}J_0(a_1,b_1)]-e^{-10/n}J_1(a_1,b_1)\} \qquad \ldots(6)$$

where

$$J_\ell(a_1,b_1)=\frac{1}{\sqrt{2\pi}}\int_{a_1}^{b_1}Z^\ell e^{-Z^2/2}dZ,\ \ell=0,1,2, \qquad \ldots(7)$$

and
$$Z=\frac{\sqrt{n}(\hat{\theta}-\theta)}{\sigma},\lambda=\frac{\sqrt{n}(\theta-\theta_0)}{\sigma},a_1=-\lambda-Z_{\alpha/2},b_1=-\lambda+Z_{\alpha/2} \qquad \ldots(8)$$

$$\text{MSE}(\theta_{PT}|\theta,R)=E(\theta_{PT}-\theta)^2$$
$$=\frac{\sigma^2}{n}\{(e^{-10/n})^2(1+\lambda^2)-\lambda^2(2e^{-10/n}-1)-(e^{-10/n})^2[J_2(a_1,b_1)+2\lambda J_1(a_1,b_1)+\lambda^2J_0(a_1,b_1)]- \qquad \ldots(9)$$
$$2e^{-10/n}\lambda[J_1(a_1,b_1)+\lambda J_0(a_1,b_1)]\}$$

3. Preliminary Test Single Stage Shrunken Estimator $\tilde{\theta}_{PT2}$

In this section, we use the following estimator to estimate the mean θ of model (1).

$$\tilde{\theta}_{PT2}=\begin{cases}(a/b)\hat{\theta}+[1-(a/b)]\theta_0 & ,\ \text{if } \hat{\theta}\in R,\\ [1-(a/b)]\hat{\theta}+(a/b)\theta_0 & ,\ \text{if } \hat{\theta}\notin R.\end{cases} \qquad \ldots(10)$$

i.e.; we put forward $\psi_1(\hat{\theta})=a/b$ and $\psi_2(\hat{\theta})=1-a/b$ in equation (3), where a and b are positive real numbers such that $a\leq b$).

The expressions for Bias and Mean Squared Error (MSE) of θ_{PT2} are respectively given as follows:-

$$\text{Bias}(\tilde{\theta}_{PT2}|\theta,R)=(\sigma/\sqrt{n})\{(2a/b-1)[J_1(a_1,b_1)-\lambda J_0(a_1,b_1)]+(a/b)\lambda\} \qquad \ldots(11)$$

and,

$$\text{MSE}(\tilde{\theta}_{PT2}|\theta,R)=(\sigma^2/n)\{(2a/b-1)[J_2(a_1,b_1)-\lambda^2J_0(a_1,b_1)]+(1-a/b)^2+(a\lambda/b)^2\} \qquad \ldots(12)$$

There is no doubt to take b=1 and find the value of (a) by minimizing the MSE($\tilde{\theta}_{PT2}|\theta,R$).

i.e. $\frac{\partial MSE(\breve{\theta}_{PT2}|\theta,R)}{\partial a}=0$.

Therefore the value of (a) becomes:

$$a^*=\frac{MSE(\hat{\theta}/\theta)-(\theta_0-\theta)Bias(\hat{\theta}/\theta)-\int_R(\hat{\theta}/\theta)^2+(\theta_0-\theta)^2]f(\hat{\theta}/\theta)d\hat{\theta}}{MSE(\hat{\theta}/\theta)-2(\theta_0-\theta)Bias(\hat{\theta}/\theta)+(\theta_0-\theta)^2}, \quad \ldots(13)$$

by simple calculation,

$$a^*=\frac{1-J_2(a_1,b_1)+\lambda^2 J_0(a_1,b_1)}{1+\lambda^2} \quad \ldots(14)$$

$$\frac{\partial^2 MSE(\breve{\theta}_{PT2}|\theta,R)}{\partial a^2}=1+\lambda^2>0$$

To be ensure that $a\in[0,1]$, we take

$$a=\begin{cases}0 & ,\text{if } a^*\le 0\\ a^* & ,\text{if } 0<a^*<1\\ 1 & ,\text{if } a^*\ge 1\end{cases}$$

we denote to the Bias Ratio of $\breve{\theta}_{PTi}$ as $B(\breve{\theta}_{PTi})$ which is defined as

$$B(\breve{\theta}_{PTi})=\frac{Bias(\breve{\theta}_{PTi}|\theta,R)}{\sigma/\sqrt{n}} \text{ for i=1,2} \quad \ldots(15)$$

See [6], [7] and [8].

The Efficiency of the proposed estimator $\breve{\theta}_{PTi\,(i=1,2)}$ relative to estimator $\hat{\theta}$ is defined as :

$$R.Eff(\breve{\theta}_{PTi}|\theta,R)=\frac{MSE(\hat{\theta})}{MSE(\breve{\theta}_{PTi}|\theta,R)} \quad , \quad i=1,2 \quad \ldots(16)$$

See [1], [6], [7] and [8].

4. Conclusions and Numerical Results

From the expressions of Bias and MSE of $\breve{\theta}_{PTi}$,i=1,2, the following could be easily seen :-

1) **i.** $B(\breve{\theta}_{PTi}|\theta,R)$ is an odd function of λ for i=1,2.

ii. $MSE(\breve{\theta}_{PTi}|\theta,R)$ is an even function of λ for i=1,2

iii. The considered estimator $\breve{\theta}_{PTi}$ is a consistent estimator of θ,

i.e; $\lim_{n\to\infty} MSE(\breve{\theta}_{PTi}|\theta,R)=0$ for i=1,2.

iv. The consider estimator $\breve{\theta}_{PTi}$ dominates $(\hat{\theta})$ with large sample size (n) in the term of MSE,

i.e.; $\lim_{n\to\infty}[MSE(\breve{\theta}_{PTi})-MSE(\hat{\theta})]\le 0$, for i=1,2

v. Practically, the consider estimator $\breve{\theta}_{PTi}$ is unbiased when $\theta=\theta_0$,

i.e.; $\lim_{\lambda\to 0} B(\breve{\theta}_{PTi}|\theta,R)=0$, for i=1,2 and for each α and λ .

2) The computations of Relative Efficiency [R.Eff($\tilde{\theta}_{PTi}$)] and Bais Ratio [B(·)][$=\sqrt{n}\,B(\tilde{\theta}_{PTi})/\sigma$] of consider estimator $\tilde{\theta}_{PTi}$ (i=1,2) were made on different constants involved in it, some of these computations are given in attached tables (1) and (2) for some samples of these constant e.g. α = 0.02, 0.01,0.05,0.1and λ = 0.0(0.1)1,2. The following numerical results from the mentioned tables leads to:-

i. Relative Efficiency of $\tilde{\theta}_{PTi}$ (i=1,2) is maximum when $\theta \approx \theta_0$, and decreases with increasing value of λ.

ii. Relative Efficiency of $\tilde{\theta}_{PTi}$ (i=1,2) is maximum when the value of α is small. i.e.; the Relative Efficiency of $\tilde{\theta}_{PTi}$ decreases with size α of the pre-test region in neighborhood of $\theta \approx \theta_0$.

iii. The Bias Ratio of $\tilde{\theta}_{PTi}$ are reasonably small when $\theta \approx \theta_0$, for i=1,2. i.e.; The Bias Ratio decreases as λ decreases.

iv. The Bias Ratio of $\tilde{\theta}_{PTi}$ decreases when α increases, for i=1,2.

v. The Effective Interval [the value of λ that makes R.Eff.(.) greater than one] using proposed estimator $\tilde{\theta}_{PTi}$ is [-1,1] for i=1,2.

vi. The estimator $\tilde{\theta}_{PT1}$ is better than the estimator $\tilde{\theta}_{PT2}$ in the sense of higher Relative efficiency for each α and λ.

vii. The Relative Efficiency of $\tilde{\theta}_{PT1}$ decreases function with increasing of (n) for each α and λ.

3) The consider estimator $\tilde{\theta}_{PTi}$(i=1,2) is better than the usual estimator ($\hat{\theta}$) and than the existing estimators, for example Thompson (1), Al-Hemyari and Al-Juboori (14) and others in terms of higher Relative Efficiency specially at $\theta \approx \theta_0$

4) From the above discussions it is obvious that by using guess point value one can improve the usual estimator. It can be noted that if the guess point θ_0 is very close to the true value of the parameter θ (i.e.; λ is approximate close to zero), the proposed estimators perform better than the usual estimator $\hat{\theta}$. If one has no confidence in the guessed value then proposed preliminary test Shrinkage estimators can be suggested. We can safely use the proposed estimators for small sample size at usual level of significance α and moderate value of shrunken weight factor $\psi(\hat{\theta})$.

References

1.Thompson,J.R., (1968), "Some Shrinkage Techniques for Estimating the Mean", J. Amer. Statist. Assoc, 63, 113-122.

2.Mehta,J.S. and Srinivasan,R., (1971), "Estimation of the Mean by Shrinkage to a Point", J. Amer. Statist. Assoc., 66, 86-90.

3.Singh, D.C., Singh, P. and Singh, P.R., (1996), Shrunken Estimator for the Scale Parameter of Classical Pareto Distribution, Microelectron Reliability, 36 (3), 435-439.

4.Al-Jubori, A.N., (2000), Preliminary Test Single Stage Shrunken Estimator for the Parameters of Simple Linear Regression Model, Ibn Al-Haitham J. for Pure and Applied Sci., 13(3), 65-73.

5.Al-Jubori, A.N., (2002), On Shrunken Estimators for the Parameters of Simple Linear Regression Model, Ibn Al-Haitham J. for Pure and Applied Sci., 15(4A), 60-67.

6.Al-Jubori, A.N., (2010), Pre-Test Single and Double Stage Shrunken Estimators for the Mean of Normal Distribution with Known Variance, Baghdad Journal for Science, 7(4), 1432-1441.

7.Al-Jubori, A.N., (2011), On Significance Test Estimator for the Shape Parameter of Generalized Rayleigh Distribution, 3rd Conf. of Computer and Mathematics College, Al-Qadisyia Univ., Al-Qadisyia, Iraq.

8.Kambo, N.S., Handa, B.R. and Al-Hemyari, Z.A., (1990), On Shrunken Estimator for Exponential Scale Parameter, Journal of Statistical Planning and Inference, 24, 87-94.

9.Mehta, J.S. and Srinivasan,R., (1971), Estimation of the Mean by Shrinkage to a Point, Jour. Amer. Statist. Assoc., 66, 86-90.

10.Pandey, B.N., (1979), On Shrinkage Estimation of Normal Population Variance, Communication in Statistics – Theory and Methods,8, 359-365

11.Prakash, G., Singh, D.C. and Singh, R.D., (2006), Some Test Estimator for the Scale Parameter of Classical Pareto Distribution, Journals of Statistical Research, 40(2), 41-54.

12.Saleh, A.K.E., (2006), Theory of Preliminary Test and Stein-Type Estimators with Application, Wiley and Sons, New York.

13.Singh, H.P. and Shukla, S.K., (2000), Estimation in the Two Parameter Weibull Distribution with Prior Information, IAPQR Transactions, 25 (2), 107-118.

14.Al-Hemyari, Z.A. and AL-Jubori A.N., (1999), "Modifical Single Stage Estimators of the Mean of Normal Population", Al-Fath J. of the College of Education for Puresci and Humanities, 3(5), 45-57.

Ibn Al-Haitham Journal for Pure and Applied Science
No. 2 Vol. 25 Year 2012
مجلة إبن الهيثم للعلوم الصرفة و التطبيقية
العدد 2 المجلد 25 السنة 2012

تقدير متوسط التوزيع الطبيعي باستعمال طريقة الاختبار الأولـي المقلصـة

سها طالب عبد الرحمن
قسم الرياضيات ¡ كلية التربية - ابن الهيثم¡ جامعة بغداد

استلم البحث في :18 كانون الاول 2011 قبل البحث في 11 كانون الثاني 2012

الخلاصة

يتعلق موضوع هذا البحث بمقدرات الاختبار الأولي المقلصة Ç المرحلة الواحدة لمتوسط التوزيع الطبيعي (q) عندما يكون التباين s^2 معلوماً وعند توافر المعلومات المسبقة (q_0) حول القيمة الحقيقية (q)، باستعمال دالة تقلص موزونة y (x̂) فضلا عن مجال الاختبار الأولي (R).

اشتقت معادلات التحيز ، متوسط مربعات الخطأ [MSE(x̂)] والكفاية النسبية [R.Eff(x̂)]، للمقدرات المقترحة. أعطيت بعض الاستنتاجات والنتائج العددية الخاصة بالمعادلات أعلاه من خلال اختيار بعض القيم للثوابت المتضمنة فيها. أجريت بعض المقارنات للمقدرات المقترحة مع المقدرات الكلاسيكية وبعض البحوث المنجزة حديثاً لبيان فائدة وأفضلية المقدرات المقترحة من حيث الكفاية النسبية.

الكلمات المفتاحية : التقدير الأولي،المقدر المقلص،دالة التقلص الموزونة،مجال الاختبار الأولي، نسبة التحيز ،متوسط مربعات الخطأ و الكفاية النسبية .