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# The Construction and Reverse Construction of the Complete Arcs in the Projective 3-Space Over Galois Field GF(2)

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## Abstract

The main purpose of this work is to find the complete arcs in the projective 3-space over Galois field GF(2), which is denoted by PG(3,2), by two methods and then we compare between the two methods.

**Keywords**: arcs, secant, quadrable.

## Introduction, [1,2]

A projective space PG(3,q) over Galois field GF(q), $q = p^m$, for some prime number p and some integer m, is a 3 – dimensional projective space.

Any point in PG(3,q) has the form of a quadrable $(x_1, x_2, x_3, x_4)$, where $x_1, x_2, x_3, x_4$ are elements in GF(q) with the exception of the quadrable consisting of four zero elements.

Two quadrables $(x_1, x_2, x_3, x_4)$ and $(y_1, y_2, y_3, y_4)$ represent the same point if there exists $\lambda$ in GF(q) \ {0} such that $(x_1, x_2, x_3, x_4) = \lambda (y_1, y_2, y_3, y_4)$, this is denoted by $(x_1, x_2, x_3, x_4) \equiv (y_1, y_2, y_3, y_4)$.

Similarly, any plane in PG(3,q) has the form of a quadrable $[x_1, x_2, x_3, x_4]$, where $x_1, x_2, x_3, x_4$ are elements in GF(q) with the exception of the quadrable consisting of four zero elements.

Two quadrables $[x_1, x_2, x_3, x_4]$ and $[y_1, y_2, y_3, y_4]$ represent the same plane if there exists $\lambda$ in GF(q)\{0} such that $[x_1, x_2, x_3, x_4] = \lambda [y_1, y_2, y_3, y_4]$, this is denoted by $[x_1, x_2, x_3, x_4] \equiv [y_1, y_2, y_3, y_4]$..

Also a point $P(x_1, x_2, x_3, x_4)$ is incident with the plane $\pi$ $[a_1, a_2, a_3, a_4]$ iff $a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 0$.

Every line in PG(3,q) contains q + 1 points and every point is on exactly q + 1 lines. Any plane in PG(3,q) contains exactly $q^2 + q + 1$ points and $q^2 + q + 1$ lines. Every point is on $q^2 + q + 1$ planes and is on $q^2 + q + 1$ lines.

Moreover PG(3,q) contains exactly $q^3 + q^2 + q + 1$ points and also contains exactly $q^3 + q^2 + q + 1$ planes.

**Definition 1: [1,3]**

A (k,n) – arc A in PG(3,q) is a set of k points such that at most n points of which lie in any plane, $n \geq 3$. n is called the degree of the (k,n) – arc.

**Definition 2: [1,3]**

In PG(3,q), if A is any (k,n) – arc, then an (n-secant) of A is a plane $\pi$ such that $|\pi \cap A| = n$.

**Definition 3: [1,3]**

Let $T_i$ be the total number of the i – secants of a (k,n) – arc A, then the type of A denoted by $(T_n, T_{n-1}, \ldots, T_0)$.

**Definition 4: [1,3]**

Let $(k_1,n)$ – arc A is of type $(T_n, \ldots, T_0)$ and $(k_2,n)$ – arc B is of type $(S_n, \ldots, S_0)$, then A and B are projectively equivalent iff $T_i = S_i$.

**Definition 5: [1,3]**

If a point N not on a (k,n)-arc A has index i iff there are exactly i(n –secants) of A through N, one can denote the number of points N of index i by $C_i$.

**Definition 6:**

If (k,n)-arc A is not contained in any (k + 1,n)-arc, then A is called a complete (k,n)-arc.

**Remark:**

From definition 5, it is concluded that the (k,n)-arc is complete iff $C_0 = 0$.

Thus the (k,n)-arc is complete iff every point of PG(3,q) lies on some n-secant of the (k,n)-arc.

**1- The Construction of Complete (k,n)-Arcs in PG(3,2)**

**1.1 The Construction of Complete (k,3)-arcs in PG(3,2):**

PG(3,q) contains 15 points and 15 planes such that each point is on 7 planes and every plane contains 7 points (see table 1).

The set A = {1, 2, 3, 4, 13} is taken which is the set of unit and reference points: 1(1,0,0,0), 2(0,1,0,0), 3(0,0,1,0), 4(0,0,0,1), 13(1,1,1,1). This set contains five points no four of them are on a plane since A intersects any plane in at most three points. Thus A is a (5,3)-arc.

A is a complete (5,3) – arc since every point of PG(3,2) not in A is on a 3-secant; that is, there are no points of index zero for A. This is equivalent to $C_0 = 0$.

**1.2 The Construction of Complete (k,4) – arcs in PG(3,2) :**

The distinct (k,4) –arcs can be constructed by adding to A in each time one point from the remaining ten points of PG(3,2) as follows:

$A_1=A\cup\{5\}$, $A_2=A\cup\{6\}$, $A_3=A\cup\{7\}$, $A_4=A\cup\{8\}$, $A_5=A\cup\{9\}$, $A_6=A\cup\{10\}$, $A_7=A\cup\{11\}$, $A_8=A\cup\{12\}$, $A_9=A\cup\{14\}$, $A_{10}=A\cup\{15\}$.

By definition 4 of projectively equivalent (k,n) – arcs, there is only one (6,4) – arc since the arcs $A_1, \ldots, A_{10}$ are projectively equivalent.

For $T_0=0$, $T_1=2$, $T_2=3$, $T_3=6$, $T_4=4$. Thus we have $B=A\cup\{5\}=\{1,2,3,4,5,13\}$ is a complete (6,4) – arc, since every point not in B is on a 4 – secant and B intersects any plane in at most 4 points, that is $C_0 = 0$.

**1.3 The Construction of Complete (k,5) – arcs in PG(3,2) :**

The arc B is a complete (6,4) – arc. The distinct (k,5) – arcs can be constructed by adding to B in each time one of the remaining nine points as follows:

$B_1$=B∪{6}, $B_2$=B∪{7}, $B_3$=B∪{8}, $B_4$=B∪{9}, $B_5$=B∪{10}, $B_6$= B∪{11}, $B_7$=B∪{12}, $B_8$=B∪{14}, $B_9$=B∪{15}.

By definition 4, there are only two projectively distinct (7,5) – arcs since the arcs $B_1$, $B_4$, $B_5$, $B_7$, $B_8$, $B_9$ are projectively equivalent, for $T_0$=0, $T_1$=1, $T_2$=2, $T_3$=5, $T_4$=6, $T_5$=1 and the arcs : $B_2$, $B_3$, $B_6$ are projectively equivalent, for : $T_0$=0, $T_1$=0, $T_2$=4, $T_3$=5, $T_4$=4, $T_5$=2. Thus we have two projectively distinct (7,5) – arcs C=B∪{6}={1,2,3,4,5,6,13}, D = B ∪ {7} ={1,2,3,4,5,7,13}.

We try to show the completeness of these arcs. Each of C and D is not complete since there exist some points of index zero.

We take the union of C and D. Then E=C∪D={1,2,3,4,5,6,7,13}, E is incomplete (8,5) – arc since there exists one point of index zero for E, which is the point (15).

We add the point (15) to E, we obtain a complete (9,5) – arc F, F=E∪{15}={1,…,7,13,15}. Thus every point not in F is on a (5 – secant) and F intersects any plane in at most 5 points.

**1.4 The Construction of Complete (k,6) – arcs in PG(3,2) :**

The arc F={1,…,7,13,15} is a complete (9,5) – arc. The distinct (k,6) – arcs can be constructed by adding to F in each time one of the remaining six points, then:

$F_1$=F∪{8}, $F_2$=F∪{9}, $F_3$=F∪{10}, $F_4$=F∪{11}, $F_5$=F∪{12}, $F_6$=F∪{14}.

By the definition 4, there are only two projectively distinct arcs since the arcs $F_1$, $F_2$, $F_5$, $F_6$ are projectively equivalent, For $T_0$=$T_1$=$T_2$=0, $T_3$=2, $T_4$=4, $T_5$=6, $T_6$=3 and the arcs $F_3$ and $F_4$ are projectively equivalent, for $T_0$=$T_1$= 2, $T_3$=2, $T_4$=4, $T_5$=7, $T_6$=2. Thus we have two projectively distinct (10,6) – arcs $G_1$={1,2,3,4,5,6,7,8,13,15}, $G_2$={1,2,3,4,5,6,7,11,13,15} each of them is incomplete since there exist some points of index zero. We take the union of $G_1$ and $G_2$. G= $G_1$∪$G_2$={1,2,3,4,5,6,7,8,11,13,15}. G is incomplete (11,6) – arc since there exists one point of index zero, which is the point (9), then H=G∪{9}={1,…,9,11,13,15}.

H is a complete (12,6) – arc, since every point not in H is on a 6 – secant and H intersects any plane in at most 6 points.

**1.5 The Construction of Complete (k,7) – arcs in PG(3,2) :**

The arc H = {1,…,9,11,13,15} is a complete (12,6) – arc. Adding all the remaining points to H, The complete (15,7) – arc can be obtained which is the maximal arc since it contains all points of PG(3,2), (see figure (1)).

**2- The Reverse Construction of Complete (k,n)-Arcs in PG(3,2):**

Complete (k,n) – arcs in PG(3,2) can be constructed by eliminating some points from the complete arcs of degree m, where $m = n + 1$, $3 \le n \le 6$, through the following steps:

**2.1 The complete (k,7) – arc in PG(3,2) :**

The projective space PG (3,2) contains 15 points and 15 planes, each plane contains exactly 7 points, then the maximal complete (k,7) – arc A exists when k = 15. This arc contains all the points of PG(3,2) since it intersects every plane in exactly 7 points and hence there arc no points of index zero for A. So A = {1, …, 15} is the complete (15,7) – arc.

**2.2 The Construction of Complete (k,6) – arc in PG(3,2) :**

A complete (k,6) – arc B is constructed from the complete (15,7) – arc A by eliminating some points from A such that:

1. B intersects any plane in at most 6 points.

**2.** every point not in B is on at least one 6 – secant of B.
The points 1, 2, 5 are eliminated from A, we obtain a complete (12,6) – arc B, since there are no points of index zero for B. B = {3, 4, 6, …, 15}.

**2.3 The Construction of Complete (k,5) – arc in PG(3,2) :**

A complete (k,5) – arc in PG (3,2) can be constructed from the complete (12,6) – arc B by eliminating some points from B, which are: 3,6,9.
Then a complete (9,5) – arc C is obtained, C = {4, 7, 8, 10, 11, 12, 13, 14, 15} since each point not in C is on at least one 5 – secant, hence there are no points of index zero for C and C intersects any plane of PG(3,2) in at most 5 points.

**2.4 The Construction of Complete (k,4) – arc in PG(3,2) :**

A complete (k,4) – arc in PG(3,2) can be constructed from the complete (9,5) – arc C by eliminating three points from C, which are the points 4, 7, 10, then a complete (6,4) – arc D is obtained, D = {8, 11, 12, 13, 14, 15} since each point not in D is on at least one 4 – secant of D and hence there are no points of index zero and D intersects each plane in at most 4 points.

**2.5 The Construction of Complete (k,3) – arc in PG(3,2) :**

A complete (k,3) – arc in PG(3,2) can be constructed from the complete (6,4) – arc D by eliminating one point from D, which is the point : 15.
A complete (5,3) – arc E is obtained, E = {8, 11, 12, 13, 14} since each point not in E is on at least one 3 – secant, hence there are no points of index zero for E and E intersects each plane in at most 3 points.
See figure (2).

## 3- Results and Conclusion

From the previous results of the two methods, we found that there is no differences between them, the numbers of the points of the complete (k,n) – arcs in the two methods given in table (2).

## References

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**Table (1):The Points $P_i$ and Planes $\pi_i$ of PG(3,2)**

| i | $P_i$ | $\pi_i$ | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | (1,0,0,0) | 2 | 3 | 4 | 6 | 7 | 10 | 12 |
| 2 | (0,1,0,0) | 1 | 3 | 4 | 7 | 9 | 14 | 15 |
| 3 | (0,0,1,0) | 1 | 2 | 4 | 5 | 8 | 10 | 15 |
| 4 | (0,0,0,1) | 1 | 2 | 3 | 5 | 6 | 9 | 11 |
| 5 | (1,1,0,0) | 3 | 4 | 5 | 7 | 8 | 11 | 13 |
| 6 | (0,1,1,0) | 1 | 4 | 6 | 11 | 12 | 13 | 15 |
| 7 | (0,0,1,1) | 1 | 2 | 5 | 7 | 12 | 13 | 14 |
| 8 | (1,1,0,1) | 3 | 5 | 10 | 11 | 12 | 14 | 15 |
| 9 | (1,0,1,0) | 2 | 4 | 9 | 10 | 11 | 13 | 14 |
| 10 | (0,1,0,1) | 1 | 3 | 8 | 9 | 10 | 12 | 13 |
| 11 | (1,1,1,0) | 4 | 5 | 6 | 8 | 9 | 12 | 14 |
| 12 | (0,1,1,1) | 1 | 6 | 7 | 8 | 10 | 11 | 14 |
| 13 | (1,1,1,1) | 5 | 6 | 7 | 9 | 10 | 13 | 15 |
| 14 | (1,0,1,1) | 2 | 7 | 8 | 9 | 11 | 12 | 15 |
| 15 | (1,0,0,1) | 2 | 3 | 6 | 8 | 13 | 14 | 15 |

**Table (2):The Maximum (k,n)-arcs in Two Methods**

| n | maximum (k,n)– arcs in the first method | maximum (k,n)– arcs in the second method |
|---|---|---|
| 3 | 5 | 5 |
| 4 | 6 | 6 |
| 5 | 9 | 9 |
| 6 | 12 | 12 |
| 7 | 15 | 15 |

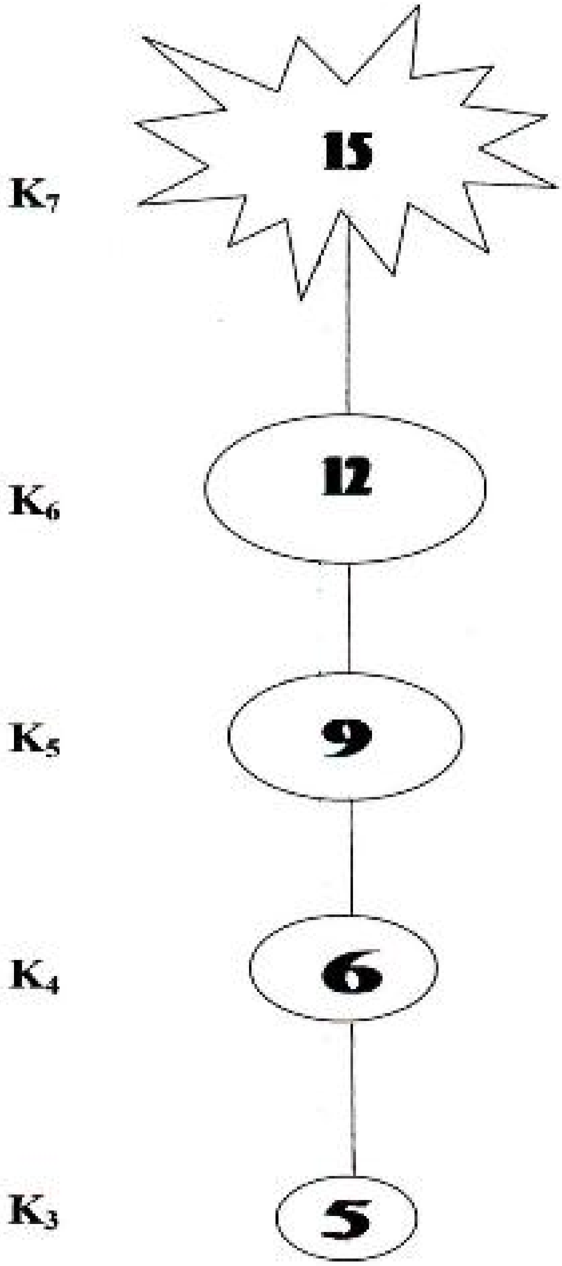


**Fig. (1):All complete ($k_n$,n) – arcs in PG(3,2), 3 ≤ n ≤ 7**

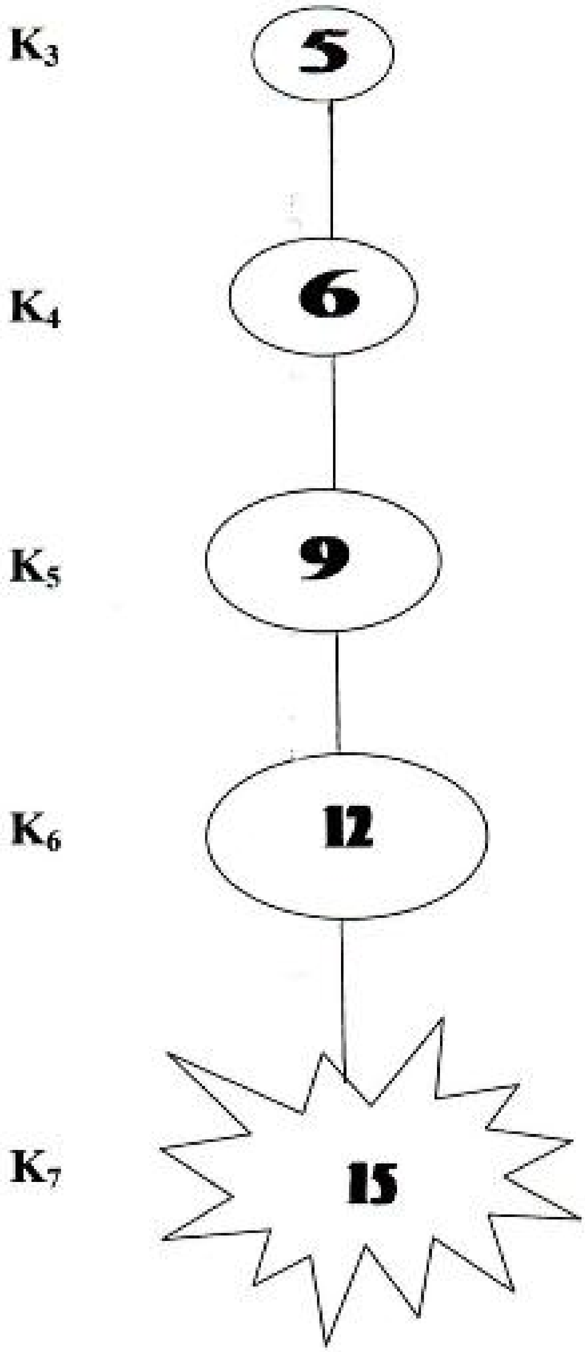


**Fig. (2):All complete ($k_n$,n) – arcs in PG(3,2), $3 \leq n \leq 7$, by reverse construction**

# البناء والبناء العكسي للأقواس الكاملة للفضاء الثلاثي الاسقاطي حول حقل كالوا GF(2)

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## الخلاصة

الهدف الاساسي من هذا البحث هو ايجاد الاقواس الكاملة في الفضاء الثلاثي الاسقاطي حول حقل كالوا GF(2)، والذي يرمز له PG(3.2)، بطريقتين ومن ثم نقارن بين الطريقتين.