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Min (Max)-CS Modules

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Abstract.

In this paper, we give a comprehensive study of min (max)-CS modules such as a closed submodule of min-CS module is min-CS. Amongst other results we show that a direct summand of min (max)-CS module is min (max)-CS module. One of interested theorems in this paper is, if R is a nonsingular ring then R is a max-CS ring if and only if R is a min-CS ring.

Key words: CS-module, min-CS module, max-CS module, uniform-CS module.

1- Introduction

Throughout the paper all rings R are commutative with identity and all R-modules are unitary. We write $A \leq M$ and $A \leq_e M$ to indicate that A is a submodule of M and A is an essential submodule of M, respectively. Recall that anR-module M is called an extending module (or, CS-module) if every submodule is essential in a direct summand of M or M is extending if and only if every closed submodule is a direct summand, [1, p.55]. In this paper definitions, notations, examples and fundamental results of min (max)-CS modules are introduced.

1.1 Definition: [2]

An R-module M is called min-CS module if every minimal closed submodule of M is a direct summand of M.

A ring R is called min-CS if it is min-CS R-module.

1.2 Definition: [2]

An R-module M is called max-CS module if every maximal closed submodule of M with nonzero annihilator is a direct summand of M.

A ring R is max-CS if it is max-CS R-module.

Recall that an R-module M is π-injective (quasi-continuous) if and only if M satisfies C1 (M is extending) and C3, where M is said to satisfy the C3 if the sum of any two direct summands of M with zero intersection is a direct summand of M. [3, p.18]

1.3 Remarks and Examples

1. Every CS-module is min-CS and max-CS.

Proof:

It follows directly by [1, p.55].

2. Every semisimple module is max-CS and min-CS.

In particular $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_{10}$,…, $\mathbb{Z}_{30}$ as a $\mathbb{Z}$-module is max-CS and min-CS.

Proof:

By [1, p.55], every semisimple module is CS. Hence the result follows by remark 1.

3. Every uniform R-module M is min-CS and max-CS.

In particular each of $\mathbb{Z}$-module $\mathbb{Z}$, $\mathbb{Z}_4$, $\mathbb{Z}_8$, $\mathbb{Z}_9$, $\mathbb{Z}_{16}$ is min-CS and max-CS.
4. If R is a semisimple ring, then every R-module M is injective, by [4, theorem 1.18, p.29]. Hence M is max (min)-CS module since every injective module is CS. By [1, p.16].

5. The $\mathbb{Z}$-module $\mathbb{Z}_{12}$ is max-CS and min-CS.
The submodules of $\mathbb{Z}_{12}$ are $<\bar{2}>, <\bar{3}>, <\bar{4}>, <\bar{6}>, <\bar{0}>$ and $\mathbb{Z}_{12}$.
Since $<3> \oplus <4> = ¢_{12}$.
So each of $<\bar{3}>$ and $<\bar{4}>$ are direct summands.
Hence they are closed submodules.
$<\bar{2}> \leq_e \mathbb{Z}_{12}$ and $<\bar{6}> \leq_e <\bar{3}>$ imply hat $<\bar{2}>$ and $<\bar{6}>$ are not closed.
Thus M is CS and so max-CS and min-CS.

6. It is easy to check that each of the $\mathbb{Z}$-modules $\mathbb{Z}_{18}$ and $\mathbb{Z}_{24}$ are min-CS and max-CS.
7. Let M be a module whose lattice of submodules is the following:

M
|
$N_1 \oplus N_2$
/ \
N_1 N_2
\ /
0

It is clear that N_1 is closed in M, but it is not a direct summand of M.
So M is not CS.
Also N_1 is a minimal closed submodule of M. Hence M is not a min-CS module.
Notice that $N_1 \oplus N_2 \leq_e M$, so it is not closed submodule of M.
It follows that N_1 is a max-closed submodule of M.
Hence M is not a max-CS module.
8. Every π-injective is min-CS and max-CS.
Proof:
It follows by the definition of π-injective module and remark 1.3 (1).

9. Let M be the $\mathbb{Z}$-module $\mathbb{Z}_8 \oplus \mathbb{Z}_2$.
M is not CS-module, since there exists a submodule $N = \{ (\bar{2},\bar{1}), (\bar{4},\bar{0}), (\bar{6},\bar{1}), (\bar{0},\bar{0}) \}$ which is closed but not a direct summand.
Moreover N is minimal closed, so M is not min-CS module.
On the other hand M is a max-CS module, since the only maximal closed submodules of M are $\mathbb{Z}_8 \oplus (\bar{0})$ and $<(\bar{3},\bar{1})>$, and $(\mathbb{Z}_8 \oplus (\bar{0})) \oplus ((\bar{0}) \oplus \mathbb{Z}_2) = M$ and $<(\bar{3},\bar{1})> \oplus ((\bar{0}) \oplus \mathbb{Z}_2) = M$. Hence we deduce that M is max-CS.

10. Let M_1 and M_2 be two R-modules such that M_1 is isomorphic to M_2 ($M_1 \cong M_2$), then M_1 min (max)-CS if and only if M_2 is min (max)-CS.
Mahmoud A.Kamal and Amany M.Menshawy in [5] gave the following:
An R-module M is called min-CS if every simple submodule of M is essential in a direct summand.

However this definition of min-CS is different from definition 1.1, since the $\mathbb{Z}$-module $M = \mathbb{Z}_8 \oplus \mathbb{Z}_2$ is not min-CS in our sense.

However it is min-CS (in sense of Kamal and Menshawy) since $\mathbb{Z}_8$ is CS, so min-CS (in sense of Kamal and Menshawy) and $\mathbb{Z}_2$ is simple, so semisimple.

Hence by applying [5, lemma 3, p.166], M is min-CS (in sense of Kamal and Menshawy).

1.4 Proposition:

Let M be an R-module, and let I be an ideal of R such that $I \subseteq annM$. M is max-CS R-module then M is max-CS (R/I)-module and the converse is true if $annM \neq annN$, for all maximal closed submodule $N \lneqq M$.

Proof:

Let N be a maximal-closed (R/I)-submodule of M and $ann_{R/I}N \neq 0_{R/I} = I$. It is easy to see that N is a maximal closed submodule of M.

Since $ann_{R/I}N \neq I = 0_{R/I}$, so there exists $r + I \in R/I$ with $r \notin I$ such that $r + I \in annN$, hence $r \neq 0$ and $rN = 0$.

Thus $ann_R N \neq 0$ and so that N is a direct summand of M.

Conversely, let N be a maximal closed R-submodule of M with $annN \neq 0$. Hence N is a maximal closed (R/I)-submodule.

Now, since $ann\, M \underset{\neq}{\subset} ann\, N$, there exists $r \in ann\, N$ and $r \notin ann\, M$.

Thus $r \notin I$; that is $0_{R/I} = I \neq r + I$ and $(r + I)N = rN = 0$.

Hence $ann_{R/I}N \neq 0_{R/I}$.

But M is a max-CS (R/ I)-module, so N is a direct summand.

1.5 Proposition:

Let M be an R-module, let I be an ideal of R such that $I \subseteq ann\, M$. Then M is min-CS R-module if and only if M is min-CS (R/ I)-module.

Proof:

It is straight forward, so it is omitted.

Recall that an R-module M is called a uniform extending (or uniform-CS) if every uniform submodule is essential in a direct summand. [1, p.55].

Al-Hazmi in [2,p.24], mentioned that min-CS and uniform-CS are equivalent concepts without proof. We shall prove this equivalence, but first we need the following lemmas.

1.6 Lemma:

Let N be a submodule of an R-module M. N is minimal closed if and only if N is uniform-closed (that is every closed submodule of N is essential in N).

Proof:

($\Rightarrow$) It is enough to prove that N is uniform

Let V, W be two nonzero submodules of N. Suppose $V \cap W = (0)$.

Hence, there exists $V' \leq N$ such that V' is a relative complement of V and hence V' is closed in N.

Since N is minimal closed of M thus N is closed in M, so by [4, proposition 1.5, p.18], V' is closed in M and $0 \neq V' \subset N$.

Thus N is not minimal closed submodule of M, which is a contradiction.

Therefore, N is uniform-closed submodule.

($\Leftarrow$) Suppose that there exists a closed submodule V of M such that $V \subseteq N$.

But N is uniform. So $V \leq_e N$.

Hence V = N, since V is closed.

Thus N is minimal closed**.**

1.7 Lemma:

If U is a uniform submodule of M such that $U \leq_e K \leq M$. Then K is uniform.

Proof:

Let V and W be two nonzero submodules of K.
Since $U \leq_e K$, $U \cap W \neq (0)$ and $U \cap V \neq (0)$.
But U is uniform submodule of M.
So $(U \cap V) \cap (U \cap W) \neq (0)$.
Hence $U \cap (V \cap W) \neq (0)$.
Thus $V \cap W \neq (0)$; that is K is uniform.

1.8 Lemma:

Let U be a submodule of an R-module M. Then U is uniform closed if and only if U is maximal uniform; that is U is maximal in the collection of uniform submodules of M.

Proof:

($\Rightarrow$) Suppose there exists a uniform submodule V of M such that $U \leq V$.
Since V is uniform so $U \leq_e V$.
But U is closed so U = V.
Thus U is a maximal-uniform submodule.
($\Leftarrow$) It is enough to show that U is closed. Suppose there exists $V \leq M$ such that $U \leq_e V$. Hence by lemma1.7, V is uniform.
Thus U = V, since U is maximal uniform, so that U is closed.

By combining lemma 1.6 and lemma 1.8, we have the following:

1.9 Corollary:

Let U be a submodule of an R-module. Then the following are equivalent:
(1) U is minimal-closed.
(2) U is uniform-closed.
(3) U is maximal-uniform.

Now, we can give the proof of the following result.

1.10 Proposition: [2, p.24]

Let M be an R-module. M is uniform-CS if and only if M is min-CS.

Proof:

($\Rightarrow$) Let U be a maximal uniform submodule of M.
Since M is uniform-CS, so U is essential in a direct summand V.
Then by lemma 1.7, V is uniform.
Hence U = V since U is a maximal uniform submodule.
Thus U is a direct summand of M, it follows that every minimal-closed is a direct summand by lemma 1.6.
So that M is a min-CS module.
($\Leftarrow$) Let U be a uniform submodule of M.
By [4, Exc.13, p.20], there exists a closed submodule V of M such that $U \leq_e V$.
Hence by lemma 1.7, V is uniform.
Thus V is a closed-uniform submodule of M, and hence by lemma 1.6, V is minimal-closed.
So that V is a direct summand of M, since M is min-CS module.
It follows that U is essential in a direct summand.
Thus M is uniform-CS.

The following result is given in [2, lemma 3.1.1, p.45], we give the details of proof.

1.11 Proposition:

Let M be an indecomposable R-module with a uniform submodule. If M is a min-CS module, then M is uniform.

Proof:

By hypothesis M has a uniform submodule, say U.
By [4, Exc.13, p.20], there exists a closed submodule K of M such that $U \leq_e K$.
Hence by lemma 1.7, K is a uniform submodule of M.
Let C = {K: K is a uniform submodule of M and $U \subseteq K$}.
Hence $C \neq \emptyset$, and so that by Zorn's lemma C has a maximal element say W.
It is clear that W is a maximal-uniform.
So it is a minimal-closed submodule of M.
Thus W is a direct summand of M, since M is a min-CS module.
Then $W \oplus V = M$ for some submodule $V \leq M$.
But M is indecomposable, hence V = (0).
Thus W = M; that is M is uniform.

1.12 Corollary:

Let M be an indecomposable R-module with a uniform submodule. Then M is min-CS module if and only if M is a uniform module.

1.13 Proposition:

Let M be an R-module. The following statements are equivalent for a module M:-
(1) M is a min-CS module.
(2) For every minimal-closed submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that A is a submodule of M_1 and M_2 is a complement of A in M.
Proof:

(1) ⇒ (2) Let A be a minimal-closed submodule in M.
Therefore A is a direct summand of M since M is a min-CS module.
That is $M = A \oplus M'$ for some submodule M' of M.
It is clear that A is a submodule of A, and it is easy to check that M' is a complement of A.
(2) ⇒ (1) To prove M is a min-CS module. Let A be a minimal-closed submodule of M.
Therefore, there is a decomposition $M = M_1 \oplus M_2$, where M_1 and M_2 are two submodules of M and A is a submodule of M_1 and M_2 is a complement of A in M.
Since M_2 is a complement of A in M, then $A \oplus M_2 \leq_e M$.
But A is a closed submodule in M and A is a submodule in M_1, therefore A is closed in M_1.
Therefore, $A \oplus M_2$ is closed in $M_1 \oplus M_2 = M$ by [4, Exc.15, p.20].
Thus $A \oplus M_2 = M$.
So that A is a direct summand in M.
Hence M is a min-CS module.

1.14 Proposition:

Let M be a min-CS R-module. If N is a closed submodule, then N is a min-CS module.
Proof:

Let U be a minimal closed submodule of N.
Since N is a closed submodule in M, then by [4, proposition 1.5, p.18], U is closed in M.
We claim that U is a minimal closed submodule of M.
To prove our assertion:
Suppose there exists a closed submodule V of M such that $V \subseteq U$.
But $V \subseteq U \subseteq N$, implies V is a closed submodule in N by [4, p.18].
But U is a minimal closed submodule of N so U = V.
Thus U is a minimal closed submodule of M, and hence U is a direct summand of M, since M is a min-CS module.
Hence $M = U \oplus W$ for some $W \leq M$.
It follows that $N = (U \oplus W) \cap N$ and by modular law, $N = U \oplus (W \cap N)$; that is U is a direct summand of N and so N is a min-CS module.

1.15 Remark:

The converse of the previous proposition is not true in general

For example:

The $\mathbb{Z}$-module $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ is not min-CS, but $N = \mathbb{Z}_8 \oplus (0) \cong \mathbb{Z}_8$ is min-CS.

1.16 Corollary:

A direct summand of min-CS module is min-CS.

Proof:

Let M be a min-CS module and let N be a direct summand of M.
Hence by [4, Exc.3, p.19], N is a closed submodule of M.
Hence, N is a min-CS module by proposition 14.

1.17 Corollary:

Let M be an R-module. If $M \oplus M$ is min-CS, then M is a min-CS module.

Proof:

It follows by corollary 1.16.

Corollary 1.16, lead us to give the following example:-

Let $M = \mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ as a $\mathbb{Z}$-module.

Thus M is not min-CS because if it is, then $N = \mathbb{Z}_8 \oplus \mathbb{Z}_2 \leq M$ is min-CS by corollary 1.16, which is a contradiction.

1.18 Remark:

The condition N is closed can not be dropped from proposition 1.14 as the following example shows:-

Let M be the $\mathbb{Z}$-module $\mathbb{Z}_{16} \oplus \mathbb{Z}_2$.

M is min-CS. Let $N = (\bar{2}) \oplus \mathbb{Z}_2$.

It is clear that $N \leq_e M$, so N is not closed in M, also N is isomorphic to $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ which is not a min-CS module.

1.19 Proposition:

Let M be a finitely generated or multiplication R-module. Then if every maximal submodule is a direct summand, then M is a max-CS module.

Proof:

Let A be a maximal closed submodule of M. Since M is a finitely generated or multiplication module.
Then by [6, theorem 2.3.11, p.28] and by [7, theorem 2.5 (1)], there exists a maximal submodule B of M such that $A \leq B$.
But B is a direct summand, by hypothesis; hence B is a closed submodule by [4, Exc.3, p.19].
It follows that A = B.
Thus A is a direct summand of M.
Hence M is a max-CS module.

The converse of the previous proposition is not true in general, as the following example shows:

1.20 Example:

$\mathbb{Z}$ as a $\mathbb{Z}$-module is a max-CS module; also $\mathbb{Z}$ is a multiplication $\mathbb{Z}$-module.

But $2\mathbb{Z}$ as a $\mathbb{Z}$-module is a maximal submodule of $\mathbb{Z}$ and it is not a direct summand.
Note that we have an analogous result to corollary 1.16, for max-CS modules, that is a direct summand of max-CS module is max-CS module, but first we prove the following:

1.21 Proposition:
Let M be an R-module, and let N be a closed submodule of M. If M is a max-CS module, then (M/N) is a max-CS R-module, provided M is not faithful.
Proof:
Let (B/N) be a maximal closed submodule in (M/N), with $ann_R(B/N) \neq 0$.
We claim that B is a maximal closed submodule in M. To prove our assertation:
First, assume that $B \leq_e L \leq M$.
But N is closed in M and $N \subseteq B \subseteq L$, so N is closed in L. Hence by [4, proposition 1.4, p.18], $N \subseteq B \leq_e L$ implies $B/N \leq_e L/N \leq M/N$.
Hence (B/N) = (L/N), since (B/N) closed in (M/N).
Thus B = L and B is closed in M.
Now, assume there exists a closed submodule B' of M such that $B \subseteq B'$, hence $N \leq B' \leq M$.
Then, by [4, Exc.16, p.20], (B'/N) is closed in (M/N) and so that $(B/N) \subseteq (B'/N)$. This implies (B/N) = (B'/N), since (B/N) is a maximal closed submodule in (M/N).
Moreover $ann_R B \supseteq ann_R M \neq 0$, so $annB \neq 0$.
Since M is max-CS then $B \oplus K = M$ for some $K \leq M$, and hence $(B/N)\oplus((K+N)/N) = (M/N)$.
It follows that (M/N) is a max-CS module.

1.22 Corollary:
A direct summand of a max-CS R-module M is a max-CS module, provided M is not faithful.
Proof:
Since N is a direct summand of M, so $M = N \oplus W$ for some $W \leq M$.
Hence (M/W) isomorphic to N by second isomorphism theorem.
But (M/W) is a max-CS by proposition 1.21.
So N is a max-CS module, by remark 1.3 (10).

1.23 Corollary:
Let M be an R-module. If $M \oplus M$ is a max-CS module. Then M is max-CS module provided M is not faithful.
Proof:
It follows by corollary 1.22.
Recall that a proper submodule N of an R-module M is called prime submodule if whenever $r \in R$, $x \in M$, $r\,x \in N$ implies $x \in N$ or $r \in [N:M]$.
An R-module M is called a prime module if $ann_R M = ann_R N$ for each submodule N of M. Equivalently, M is called a prime module if (0) is a prime submodule of M, [8].

1.24 Corollary:
Let M be a not faithful prime max-CS R-module. If $f : M \longrightarrow M'$ is an epimorphism, then M' is a max-CS module.
Proof:
M is a max-CS module.
Then $(M/\ker f) \cong M'$, by the 1st fundamental theorem.
We claim that ker f is clsed submodule in M.
To show this, let $\ker f \lneqq_e L \leq M$.
Assume there exists $0 \neq x \in L$ such that $x \notin \ker f$.

Then there exists $0 \neq r \in R$ such that $0 \neq r x \in \ker f$.
Hence $r \cdot f(x) = 0$.
Since M is a prime module, so either $f(x) = 0$ or $r \in annM$.
But $f(x) \neq 0$ since $x \notin \ker f$, hence $r \in annM$ and this implies $r x = 0$ which is a contradiction.
Thus ker f = L and so that ker f is a closed submodule of M.
Since annM $\neq 0$ by hypothesis. Then by proposition 1.21, (M/ker f) is a max-CS module and hence by remark 1.3 (10) M' is a max-CS module.

1.25 Corollary:

Let M be a max-CS not faithful R-module. If N is a prime submodule of M and $[N \underset{R}{:} M] = ann_R M$. Then (M/N) is a max-CS R-module.

Proof:

Since N is a prime R-submodule of M, then (M/N) is a torsion free (R/[N:M])-module, [8, p.61-69].
Hence (M/N) is a torsion free (R/annM)-module, and so that N is closed (R/annM)-module of M by [9, remark 3.3,p.48].
It follows that N is closed R-submodule of M. Then by proposition 1.15, (M/N) is a max-CS R-module.

1.26 Corollary:

If M is a max-CS not faithful R-module and N is a submodule of M, such that (M/N) is torsion free, then (M/N) is max-CS.

Proof:

Since (M/N) is torsion free R-module. Then N is closed submodule in M by [9, remark 3.3, p.48]. Hence the result follows by proposition 1.21.

Now we have the following note for min-CS modules.

1.27 Remark:

The homomorphic image of min-CS need not be a min-CS, as the following examples show.

1.28 Examples:

(1) Let M be the $\mathbb{Z}$-module $\mathbb{Z}\oplus\mathbb{Z}$. It is clear that M is a min-CS module. Let $N = 8\mathbb{Z}\oplus 2\mathbb{Z}$, and let $\pi: M \longrightarrow (M/N)$ be the natural projection.
Since (M/N) is isomorphic to $\mathbb{Z}_8\oplus\mathbb{Z}_2$, so $(M/N) = \pi(M)$ is not min-CS module.

(2) Let M be the $\mathbb{Z}$-module $\mathbb{Z}\oplus\mathbb{Z}_2$. M is min-CS. Let $N = 8\mathbb{Z}\oplus(\bar{0})$, $N \leq_e \mathbb{Z}\oplus(\bar{0})$, but $(M/N) = \mathbb{Z}_8\oplus\mathbb{Z}_2$ is not min-CS.

(3) Let $M = \mathbb{Z}_8\oplus\mathbb{Z}$ be a $\mathbb{Z}$-module, M is a CS-module. So M is a min-CS module. Let $N = (\bar{0}) \oplus (\bar{2})$. (M/N) is isomorphic to $\mathbb{Z}_8\oplus\mathbb{Z}_2$ which is not min-CS module.

1.29 Theorem:

Let M be an R-module. If M is a faithful, finitely generating and multiplication R-module, then M is a max-CS module if and only if R is a max-CS ring.

Proof:

($\Rightarrow$) If M is a max-CS R-module. Let I be a maximal closed ideal of R such that ann I$\neq$0.
We claim that N = IM is a maximal closed R-submodule of M.
First of all N = [IM:M]M since M is multiplication and by [10, proposition 3.31] [IM:M] is a closed ideal of R.
But M is finitely generating faithful multiplication, then by [7, theorem 3.1] I = [IM:M].

Thus by [10, proposition 3.31, ch.3], N = [IM:M]M is a closed submodule of M.
Now, to prove that N is a maximal submodule of M.
Suppose there exists a closed submodule W of M such that $N \subseteq W$.
Hence $N = IM = [N:M]M \subseteq [W:M]M = W$ and by [7, theorem 3.1], $I = [N:M] \subseteq [W:M]$.
On the other hand by [10, proposition 3.31, chapter three], I = [N:M] and [W:M] are closed ideals of R.
Hence I = [W:M], since I is a maximal closed ideal of R.
It follows N = W and N is maximal closed submodule of M.
But M is faithful multiplication, so $ann_R N = ann_R I \neq 0$.
Thus N is a direct summand of M.
So that $N \oplus L = M$ for some $L \leq M$.
But N = IM, L = [L:M]M, so that $IM \oplus [L:M]M = M$ and hence $(I \oplus [L:M])M = M$. But by [7, theorem 1.6] $(I \cap [L:M])M = IM \cap [L:M]M = (0)$, and so $(I \cap [L:M])M = 0$; that is $(I \cap [L:M]) \subseteq annM = (0)$. Thus $I \cap [L:M] = (0)$, so that $M = RM = (I \oplus [L:M])M$ and hence $R = I \oplus [W:M]$ by [7, theorem 6.1].
($\Leftarrow$) Let N be a maximal closed submodule of M with $annN \neq 0$.
Since N is closed, then by [10, proposition 3.1] [N:M] is closed in R.
We claim that [N:M] is a maximal closed ideal in R.
To show this, assume J is a closed ideal in R such that $[N:M] \subseteq J$, hence $N = [N:M]M \subseteq JM$.
But JM = [JM:M]M and by [7, theorem 6.1] [JM:M] = J and by [10, proposition 3.1] JM is a closed submodule of M.
It follows that N = [N:M]M = JM, since N is a maximal closed submodule of M. Then by [7, theorem 3.1], [N:M] = J; that [N:M] is a maximal closed.
But annN = ann[N:M] since M is faithful multiplication.
Thus [N:M] is a maximal closed ideal of R, with $ann_R[N:M] \neq 0$.
On the other hand R is a max-CS ring, so [N:M] is a direct summand of R.
Thus $R = [N:M] \oplus T$ where T is an ideal of R, and hence

$$M = RM = ([N:M] \oplus T)M$$
$$= [N:M]M \oplus TM.$$

But by [7, theorem 1.6], $[N:M]M \cap TM = ([N:M] \cap T)M = 0M = 0$
So $M = [N:M]M \oplus TM$; that is $M = N \oplus TM$.

By the same argument of proof of theorem 1.29, we have the following:

1.30 Proposition:

Let M be a faithful, multiplication and finitely generated R-module, then M is a min-CS ring if and only if R is min-CS ring.

Next we have for nonsingular rings, the concepts min-CS ring and max-CS ring are equivalent. But first we need the following which is given in [2, lemma 2.1.1, p.31]. We give the details of proof.

1.31 Lemma:

For a ring R, a complement of minimal (maximal) closed ideal of R is a maximal (minimal) closed ideal of R.

Proof:

($\Rightarrow$) Let I be a minimal closed ideal of R. Let J be the relative complement of I. Then by [4, proposition 1.3, p.17], $(I \oplus J) \leq_e R$, and J is closed in R, by [4, proposition 1.4, p.18].
Now, we shall show that J is a maximal closed ideal in R.
Assume that there exists a closed ideal J* in R, such that $J \subsetneq J^* \subsetneq R$.
It follows that $J^* \cap I \neq (0)$ because J is the largest ideal in R such that $I \cap J = (0)$.

On the other hand I is a minimal closed ideal in R, so it is uniform closed by lemma 1.6. Hence $J^* \cap I \leq_e I$. But $J \leq_e I$, hence
$(J^* \cap I) \oplus J \leq_e I \oplus J \leq_e R$ and so that $(J^* \cap I) \oplus J \leq_e R$, by [4, proposition 1.1(a), p.16].
But $J^* \cap I \leq J^*$ and $J < J^*$, hence $(J^* \cap I) \oplus J \leq J^*$.
Thus $J^* \leq_e R$, which is a contradiction.
($\Leftarrow$) If T is a maximal closed ideal of R and V its complement.
To prove V is minimal closed in R, it is enough to show that V is uniform by lemma 1.6.
Assume $A \cap B = (0)$ for some two nonzero ideals A and B of V.
Now, there exists a closed ideal I in R such that $A \oplus T \leq_e I$, by [4, Exc.13, p.20].
Hence $T \underset{\neq}{\subset} I$ which is a contradiction, since I is closed in R and T is a maximal closed ideal in R. Thus V is uniform closed, that is V is minimal closed.

1.32 Note:

By a similar argument of proof of lemma 1.31, we get analogous results for submodules.

Recall that a ring R is semiprime if for each $x \in R$, $x^2 = 0$, then $x = 0$, [4, p.2].

Now, we can give the following theorem:

1.33 Theorem:

Let R be a nonsingular ring. Then R is max-CS if and only if R is min-CS.

Proof:

($\Rightarrow$) If R is a max-CS ring. Let I be a minimal closed ideal in R.
By [4, theorem 2.38, p.65], I = ann annI, hence annI $\neq$ 0, but R is nonsingular, so R is semiprime by [4, proposition 1.27(b), p.35], which implies $I \cap \text{ann}I = 0$.
Let J be the relative complement of I, so by lemma 1.31, J is a maximal closed ideal in R.
Also by [4, theorem 2.38, p.65], J = ann annJ, so annJ $\neq$ 0.
Therefore J is a direct summand of R since R is max-CS.
It follows that J = <e> for some idempotent e in R.
On the other hand, $IJ \subseteq I \cap J = (0)$, implies that $J \subseteq \text{ann}I$.
But $I \cap \text{ann}I = (0)$, so J = annI since J is the largest ideal in R such that $I \cap J = (0)$.
Thus annJ = ann annI = I and so $< 1 - e > = I$.
It follows that I is a direct summand of R.
($\Leftarrow$) If R is a min-CS ring.
Let I be a maximal closed ideal in R, with annI $\neq$ 0.
By [4, theorem 2.38, p.65], I = ann annI. Let J be a relative complement of I. So J is a minimal closed ideal of R, by Lemma 1.31.
Hence J is a direct summand of R, since R is min-CS.
Thus $J = < e >$ for some idempotent $e \in R$.
But $IJ \subseteq I \cap J = (0)$, so $I \subseteq \text{ann } J = \text{ann} <e> = <1 - e>$.
But $<1 - e>$ is closed ideal of R, since it is a direct summand of R.
It follows that $I = <1 - e>$ because I is a maximal closed ideal of R.
Thus I is a direct summand of R.

Recall that, a ring R is called semihereditary if every finitely generating ideal of R is projective, [4, p.10].

1.34 Corollary:

Let R be a semiheriditary ring. Then R is a min-CS ring if and only if R is a max-CS ring.

Proof:

Since R is a semiheriditary ring. Then R is a nonsingular ring, [4, p.36]. So we get the result by Theorem 1.33.

Recall that a ring R is called regular (Von Neumann) if for every $a \in$R there is an $x \in$ R such that $a\,x\,a = a$, that is $a = a^2 x$, [4, p.10].

1.35 Corollary:

If R is a regular ring. Then R is min-CS if and only if R is a max-CS.

The following lemma is needed for the following corollary.

1.36 Lemma: [4, Exc.5, p.36]

For a commutative ring, R/Z(R) is a nonsingular ring.

Proof:

Suppose there exists $\bar{x} = x + Z(R) \in Z(R/Z(R))$ and $x \notin Z(R)$, so $x \neq 0$ and ann(x) is not essential in R, also $ann_{R/Z(R)}(x + Z(R)) \leq_e R/Z(R)$.

Then for each I/Z(R) with $I \underset{\neq}{\supset} Z(R)$, $(I/Z(R)) \cap ann_{R/Z(R)}(x+Z(R)) \neq O_{R/Z(R)} = Z(R)$. So there exists $a + Z(R) \in ann_{R/Z(R)}(x + Z(R))$, $a \in$ I such that $a \notin Z(R)$.

Then $a \neq 0$.

Thus $ax + Z(R) = Z(R)$, $ann_R(a)$ is not essential in R and $a \in$ I.

Hence $ax \in Z(R)$ and $ann_R(a)$ is not essential in R. So that $ann(ax) \leq_e R$ and ann(a) is not essential in R.

Now, since $ann(ax) \leq_e R$, then for each nonzero ideal J of R, $J \cap ann(ax) \neq 0$.

Hence there exists $y \in ann(ax)$ and $y \neq 0$. Thus $yax = 0$.

(1) If $ya = 0$ then $0 \neq y \in ann(a)$ thus $0 \neq y \in J \cap ann(a)$ so that $ann(a) \leq_e R$ which is a contradiction.

(2) If $ya \neq 0$, then $0 \neq ya \in ann(x) \cap J$. Hence $ann(x) \leq_e R$ which is a contradiction.

(3) If $yx = 0$ then $0 \neq y \in ann(x)$, so that $0 \neq y \in ann(x) \cap J$. Thus $ann(x) \leq_e R$ which is a contradiction.

(4) If $yx \neq 0$, then $0 \neq yx \in ann(a) \cap J$, so that $ann(a) \leq_e R$ which is a contradiction.

(5) If $ax = 0$ then $0 \neq a \in ann(x) \cap I$, that is $ann(x) \leq_e R$ which is a contradiction.

Thus our assumption is false and so $Z(R/Z(R)) = Z(R) = O_{R/Z(R)}$.

1.37 Corollary

Let R be a ring. Then R/Z(R) is min-CS if and only if R/Z(R) is max-CS.

Proof:

By lemma 1.36 R/Z(R) is nonsingular, so the result follows by theorem 1.33.

Before we give the following corollary, we need the following lemma.

1.38 Lemma: [4, Exc.13, p.37]

If R is a nonsingular ring, then $R[x_1, x_2, \ldots, x_n]$ is a nonsingular ring.

Proof:

First we shall prove that R[x] is a nonsingular.

Let $f(x) \in Z(R[x])$, then $ann_{R[x]} f(x) \leq_e R[x]$.

Assume $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, where $a_i \in$ R, for each i = 0, 1, …, n.

$ann_{R[x]} f(x) = ann_{R[x]} a_0 \cap ann_{R[x]} a_1 x \cap \ldots \cap ann_{R[x]} a_n x^n \leq_e R[x]$.

So, $ann_{R[x]} a_0 \leq_e R[x]$, $ann_{R[x]} a_1 x \leq_e R[x]$, …, $ann_{R[x]} a_n x^n \leq_e R[x]$.

We claim that $ann_R(a_i) \leq_e R$, for all i = 0, 1, …, n.

To prove this. Suppose there exists $J \leq R$ and $J \neq 0$ such that $ann_R a_0 \cap J = 0$.

$J \neq 0$ so J[x] is an ideal in R[x] and $J[x] \neq 0$. Hence $J[x] \cap ann_{R[x]} a_0 \neq 0$.

Let $0 \neq g(x) \in ann_{R[x]} a_0$ and $g(x) \in J[x]$, such that $g(x) = b_0 + b_1 x + \ldots + b_m x^m$, where $b_i \in$ J, for all i = 0, 1, …, m and there exists $k \in \{0,1,\ldots,m\}$ with $b_k \neq 0$. Since $g(x) \in ann_{R[x]}(a)$, then $(b_0 + b_1 x + \ldots + b_k x^k + \ldots + b_m x^m) a_0 = 0$.

Thus $b_0\, a_0 + b_1\, a_0\, x + \ldots + b_k\, a_0\, x^k + \ldots + b_m\, a_0\, x^m = 0$
which implies $b_0\, a_0 = 0$, $b_1\, a_0 = 0$, …, $b_k\, a_0 = 0$, …, $b_m\, a_0 = 0$. But $b_k\, a_0 = 0$ and $0 \neq b_k \in J$ implies that $J \cap \text{ann}_R a_0 \neq 0$, which is a contradiction with our assumption. Hence $\text{ann}_R a_0 \leq_e R$, so that $a_0 \in Z(R)$. But R is nonsingular, hence $a_0 = 0$. By the same way we can prove that each of $a_1, \ldots, a_m$ belongs to Z(R)=(0). Thus f(x) = 0 and Z(R[x]) = (0); that R[x] is nonsingular.
Then by induction $R[x_1, x_2, \ldots, x_n]$ is a nonsingular ring.

1.39 Corollary:
Let R be a nonsingular ring. Then $R[x_1, x_2, \ldots, x_n]$ is min-CS if and only if $R[x_1, x_2, \ldots, x_n]$ is a max-CS ring.
Proof:
By lemma 38 if R is a nonsingular ring, then $R[x_1, x_2, \ldots, x_n]$ is a nonsingular ring. Hence the result follows by theorem 33.

1.40 Note:
A direct sum of min (max)-CS modules will be discussed in another paper.

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أصغر (أعظم) مقاسات التوسع

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الخلاصة

في هذا البحث نعطي دراسة واسعة لأصغر (أعظم) مقاسات التوسع مثل المقاس الجزئي المغلق من أصغر مقاس توسع هو أصغر مقاس توسع. ومن بين النتائج الاخرى نستعرض أنه مركبة المجموع المباشر لأصغر (أعظم) مقاس توسع هو أصغر (أعظم) مقاس توسع وكذلك اذا كانت الحلقة R غير مفرده فان الحلقة R أعظم حلقة توسع إذا وفقط إذا كانت الحلقة R أصغر حلقة توسع والتي هي واحدة من المبرهنات المهمة في هذا البحث.

الكلمات المفتاحية: مقاس توسع، أصغر مقاس توسع، أعظم مقاس توسع، مقاس توسع منتظم.