Ibn Al-Haitham Journal for Pure and Applied Science | مجلة إبن الهيثم للعلوم الصرفة و التطبيقية
No. 1 Vol. 25 Year 2012 | 2012 السنة 25 المجلد 1 العدد

On Weakly Quasi-Prime Module

M. A. Hassin
Department of Mathematical, College of Basic Education, University of Al-Mustansriyah
Received in: 3 April 2011 Accepted in: 18 October 2011

Abstract

In this work we shall introduce the concept of weakly quasi-prime modules and give some properties of this type of modules.

Key words: Prime module, quasi-prime module, weakly quasi-prime module.

1- Introduction

Let R be a commutative ring with unity, and let M be an R-module, we introduce that an R-module M is called weakly quasi–prime module if $ann_R M = ann_R rM$ for every $r \notin ann_R M$, where $ann_R M = \{r: r \in R \text{ and } rM = 0\}$.

The main purpose of this work is to investigate the properties of weakly quasi-prime modules, and we give several characterizations of weakly quasi-prime modules. Recall that an R-module is called prime if $ann_R M = ann_R N$ for every non-zero submodule N of M and $ann_R M = \{r: r \in R \text{ and } rM = 0\}$, [1].

A submodule N of M is said to be prime if $a\,m \in N$ for $a \in R$, $m \in M$, then either $m \in N$ or $a \in [N:M]$ where $[N:M] = \{r: r \in R, rM \subseteq N\}$, [1], [2].

It was shown that in [1] M is prime module iff (0) is prime submodule.

The concept of quasi-prime module is introduced in [3] where an R-module M is quasi-prime module if $ann_R N$ is prime ideal for every nonzero submodule N of M. If M is quasi-prime module then $ann_R M = ann_R rM$ $\forall\, r \notin ann_R M$, [3]. But the converse is not true for example:

Let $M = Z_{p^\infty}$ as Z-module is not quasi-prime module since if $N = <1/p^2 + z> \leq Z_{p^\infty}$. So $ann_R N = p^2 z$ is not prime ideal in Z.

But $ann Z_{p^\infty} = 0$ and $\forall\, r \neq 0$, let $a \in ann\, r Z_{p^\infty}$ so $a\, r Z_{p^\infty} = 0$, so $a\, r \in ann Z_{p^\infty}$.

$a\, r = 0$, but $r \neq 0$ so $a = 0$ so $ann\, r Z_{p^\infty} = 0$. Then $ann\, Z_{p^\infty} = ann\, r Z_{p^\infty}$.

2- Weakly Quasi-Prime Module

In this section we introduce the concept of weakly quasi-prime module and give several results about it.

2.1 Definition:

An R-module M is called weakly quasi-prime module (briefly W.q.p) if $ann_R M = ann_R rM$ for every $r \notin ann_R M$.

Recall that if R is an integral domain, an R-module M is said to be divisible iff $rM = M$ for every nonzero element r in R, [4,p.35].

2.2 Examples and Remarks:

1. If M is divisible over integral domain then M is W.q.p.
2. Every quasi-prime is W.q.p but the converse is not true (see the example in the introduction).
3. Z as Z-module is W.q.p module since $ann_R Z = 0 = ann_R\, rZ$, $\forall$ $r \notin ann_R Z$.
4. Z_4 as Z-module is not W.q.p module Since $ann_R Z_4 = 4Z$ and $ann_R 2Z = ann_R(\bar{2}) = 2Z$. Thus Z_4 as Z-module is not W.q.p module.
5. Z_6 as Z-module is not W.q.p module since $annZ_6 = 6Z$ and $ann2Z_6 = ann(\bar{2}) = 3Z$, so $annZ_6 \neq ann\ 2Z_6$.
6. Z_n as Z-module is W.q.p module iff n is prime.
7. Let $M = Z \oplus Z_p$; p is prime number is W.q.p module since $annM = ann\ rM = 0$ for each $r \notin ann(Z \oplus Z_p)$.
8. Z_{p^∞} is W.q.p module since $ann\ Z_{p^\infty} = ann\ r Z_{p^\infty} = 0$.

2.3 Note:

Let M be W.q.p over integral domain in R. Then every divisible submodule of W.q.p module. Recall that a proper submodule N of M is called semi-prime submodule if every $r \in R$, $x \in M$, $k \in Z_+$, such that $r^k x \in N$, then $rx \in N$, [4,p.50].

2.4 Proposition:

Let M be divisible and (0) submodule of M is semi-prime submodule, then the following statements are equivalent

1. M is prime module,
2. M is q.p module,
3. M is W.q.p module.

Proof :(1) → (2), by [2,p10]

(2) → (3), by [2,p20]

(3) → (1) To prove M is prime module, i.e. to show that (0) is prime submodule.

Let $rm = 0$, $r \in R$, $m \in M$, to prove either $m = 0$ or $r \in ann_R M$. Suppose $r \notin ann_R M$, so we must prove that $m = 0$. Since $r \notin ann_R M$, $rM \neq 0$. Hence $rM = M$, because M is divisible. Thus $m = rm_1$ for some $m_1 \in M$. Since $rm = r(rm_1) = 0$, that is $r^2 m_1 = 0$ which implies that $rm_1 = 0$, since (0) submodule of M is semi-prime. Thus $m = 0$.

2.5 Remark:

The condition in proposition 2.4 is necessary as the following example shows:

Z_{p^∞} is not q.p since if $N = \frac{1}{p^2} + Z$ then $ann\ N = p^2 Z$ is not prime ideal, but Z_{p^∞} is W.q.p module (see the example in the introduction).

2.6 Theorem:

Let M be a module over an integral domain R and every submodule of M is divisible then $ann\ (rm) = ann\ (m)$, for each $r \notin ann\ (m)$.

Proof: Since $(rm) \subseteq (m)$, so

$ann(m) \subseteq ann\ (rm)$ …(1)

To prove $ann\ (rm) \subseteq ann\ (m)$

Let $x \in ann\ (rm)$ so $x\ (rm) = 0$. Since every submodule of M is divisible, $(rm) = (m)$ and so $xm = 0$ which implies $x \in ann\ (m)$. Thus

$ann(rm) \subseteq ann(m)$ …(2)

From (1) and (2), we have $ann\ (m) = ann(rm)$, for each $r \notin ann\ (m)$.

Recall that an R-module M is called multiplication R-module if for every submodule N of M, there exists an ideal I of R such that IM = N.

2.7 Theorem:

Let M be multiplication W.q.p R-module. Then every submodule of M is W,.q.p module.

Proof: Let N be submodule of M, since M is multiplication R-module, so N = IM; I be ideal of ring R. To prove N is W.q.p module.

To prove $ann_R N = ann_R rN$, $\forall$ $r \notin ann_R N$ since $rN \subseteq N$ so

$ann_R N \subseteq ann_R rN$ …(1)

To prove $ann_R rN \subseteq ann_R N$. Let $x \in ann_R rN$ so $xrN = 0$. Since M is multiplication so there exists an ideal I of R such that N = IM. Thus $xrIM = 0$; that is $xI \subseteq ann_R rM = ann_R M$, hence $xIM = 0$; so $xN = 0$ which implies $x \in ann_R N$. Thus

$ann_R rN \subseteq ann_R N$ …(2)

From (1) and (2) we have $ann_R N = ann_R rN$ so N is W.q.p module.

2.8 Proposition**:**

Let M be cyclic W.q.p R-module. Then M is q.p module.

Proof: Let M be cyclic so there exist $x \in M$; $M = (x)$, let $y \in M$, to prove $ann_R y$ is prime ideal, so $y = rx$; $r \in R$, let a, $b \in ann_R y$, to prove either $a \in ann_R y$ or $b \in ann_R y$. Since $ab \in ann_R y = ann_R rx$, so $abrx = 0$. Suppose $b \notin ann_R y = ann_R rx$, i.e $brx \neq 0$, so $ab \in ann_R(rx) = ann_R(x)$, since M is W.q.p module, so $abx = 0$ which implies that $a \in ann_R bx = ann_R(x)$ (since M is W.q.p). Thus $ax = 0$ which implies $rax = r.0 = 0$ so $a \in ann\,(rx)$ which means $a \in ann_R y$.

2.9 Theorem:

Let M be cyclic R-module then the following statements are equivalent

1. M is prime module
2. $ann_R M = ann_R IM$; $I \nsubseteq ann_R N$
3. M is W.q.p module.

Proof: To prove (1) → (2)

It is clear by definition of prime submodules.

(2) → (3) it is obvious.

To prove (3) → (1), to prove M is prime module.

By proposition (2.8) we have M is q.p module which implies that $ann_R M$ is prime ideal, see [3,p.14] and by [3,p.8] we get M is a prime module.

2.10 Theorem**:**

The direct sum of two W.q.p R-module is also W.q.p R-module.

Proof: Let $M = M_1 \oplus M_2$ where M_1 and M_2 are two W.q.p module, to prove M is W.q.p module, i.e to prove $ann_R M = ann_R rM$, for all $r \notin ann_R M$.

$ann_R rM = ann_R r(M_1 \oplus M_2)$

$= ann_R(rM_1 \oplus rM_2)$, see [2, p.80]

$= ann_R rM_1 \cap ann_R rM_2$, see [2, p.83]

$= ann_R M_1 \cap ann_R M_2$, since M_1 and M_2 are W.q.p

$= ann_R(M_1 \oplus M_2)$

$= ann_R M$

2.11 Corollary:

Let M be an R-module if M is W.q.p module then for any positive integer n, M^n is W.q.p module where M^n is the direct sum of n copies of M.

2.12 Remark:

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A direct summand of W.q.p module is need not be W.q.p module.
For example: Let $M = Z \oplus Z_4$ so $ann_R M = ann_R rM$ $\forall$ $r \notin ann_R M$. But Z_4 is not W.q.p module, (see remarks and examples (2.2(4)).

2.13 Theorem:

Let M_1 ; M_2 then M_1 is W.q.p iff M_2 is W.q.p.

Proof: $\Rightarrow$ Let f: $M_1 \rightarrow M_2$ be 1-1 and onto and homomorphisim and M_2 is W.q.p. To prove $M_1 = f^{-1}(M_2)$ is W.q.p module, that is to prove $ann_R r f^{-1}(M_2) \subseteq ann_R f^{-1}(M_2)$; $r \notin ann_R f^{-1}(M_2)$, let $x \in ann_R\, r\, f^{-1}(M_2)$ so $xrf^{-1}(M_2) = 0$ and since $f^{-1}$is homomorphisim so $f^{-1}(xrM_2) = f^{1}(0)$ and since f^{-1} is 1-1 so $xrM_2 = 0$ which mean $x \in ann_R rM_2$ but M_2 is W.q.p module and $r \notin ann_R M_2$ then $xM_2 = 0$ which implies $f^{-1}(xM_2) = f^{-1}$ (0), but f^{-1} is homomorphisim so $x\, f^{-1}(M_2) = 0$ implies $x \in ann_R f^{-1}(M_2)$ so

$ann_R r\, f^{-1}(M_2) \subseteq ann_R\, f^{-1}(M_2)$ …(1)

and since $r\, f^{-1}(M_2) \subseteq f^{-1}(M_2)$, so

$ann_R f^{-1}(M_2) \subseteq ann_R r\, f^{-1}(M_2)$ …(2)

From (1) and (2) we have $ann_R f^{-1}(M_2) = ann_R r f^{-1}(M_2)$. So $f^{-1}(M_2)$ is W.q.p module.

$\Leftarrow$ clearly.

2.14 Note:

The condition "isomorphism" in theorem 2.13 is necessary as the following example shows

Example: Let π: $Z \longrightarrow Z/(4)$; Z_4, where Z is W.q.p, but Z_4 is not W.q.p.

It is known that, if M is an R-module and I is an ideal of R which is contained in annRM then M is R/I-module, by taking $(r + 1)x = rx$ $\forall x \in M$, $r \in R$, see [5,p.40].

Now, we give the following result.

2.15 Theorem:

Let M be an R-module and let I be an ideal of R, which is contained in $ann_R M$. Then M is W.q.p R-module iff M is W.q.p R/I-module.

Proof: $\Rightarrow$ To prove M is W.q.p R/I-module, i.e. to prove $ann_{R/I} M = ann_{R/I}(r + 1)M$. Since $(r + 1)M \subseteq M$ so

$ann_{R/I} M \subseteq ann_{R/I}(r + 1)M$ …(1)

To prove $ann_{R/I}(r + 1)M \subseteq ann_{R/I} M$

Let $x \in ann_{R/I}(r + 1)M$ so $x(r + 1)M = 0$, which implies $(xr + 1)M = 0$ so $(xr)M = 0$ (by definition), so $x \in ann_R rM = ann_R M$ (since M is W.q.p R-module).

$x \in ann_{R/I} M$ (since $I \subseteq ann_{R/I} M$), so

$ann_{R/I}(r + 1)M \subseteq ann_{R/I} M$ …(2)

From (1) and (2) we have $ann_{R/I} M = ann_{R/I}(r + 1)M$.

$\Leftarrow$ If M is W.q.p R/I-module then M is W.q.p R-module, i.e. to prove $ann_R M = ann_R rM$, $\forall$ $r \notin ann_R M$. Since $rM \subseteq M$ so

$ann_R M \subseteq ann_R rM$ …(1)

To prove $ann_R rM \subseteq ann_R M$

Let $x \in ann_R rM$ so $(xr)M = 0$ implies that $(xr + 1)M = 0$, so $x(r + 1)M = 0$, hence $x \in ann_{R/I}(r + 1)M = ann_{R/I} M$ (since M is W.q.p R/I-module). Thus $x \in ann_{R/I} M$, which implies that $x \in ann_R M$ (since $I \subseteq ann_R M$), so

$ann_R rM \subseteq ann_R M$ …(2)

From (1) and (2) we have $ann_R M = ann_R rM$.
So M is W.q.p module.

Recall that a subset S of a ring R is called multiplicatively closed if $1 \in S$ and $a \cdot b \in S$ for every $a, b \in S$. We know that every proper ideal P in R is prime if and only if R-P is multiplicatively closed, see [4,p.42].

Let M be a module on the ring R and S be a multiplicatively closed on R such that $S \neq 0$ and let R_S be the set of all fractional r/s where $r \in R$ and $s \in S$ and M_S be the set of all fractional x/s where $x \in M$, $s \in S$; $x_1/s_1 = x_2/s_2$ if and only if there exists $t \in S$ such that $t(s_1x_2 - s_2x_1) = 0$. So, can make M_S into R_S-module by setting $x/s + y/t = (tx + sy)/st$, $r/t \cdot x/s = rx/ts$ for every $x, y \in M$ and for every $r \in R$, $s, t \in S$. If S = R-P where P is a prime ideal we use M_P instead of M_S and R_P instead of R_S. A ring in which there is only one maximal ideal is called local ring, see [4,p.50], hence R_P is often called the localization of R, similar M_P is the localization of M at P. So we can define the two maps $\psi: R \longrightarrow R_S$, such that $\psi(r) = r/1$, $\forall r \in R$, $\phi: M \longrightarrow M_S$, such that $\phi(m) = m/1$, $\forall m \in M$, see [5,p.69]. Through this paper $S^{-1}R$ and $S^{-1}M$ represent R_S and M_S respectively.

2.16 Proposition:

Let M be W.q.p R-module then $S^{-1}M$ is W.q.p $S^{-1}R$-module for each multiplicatively closed set S of R.

Proof**:** To prove $ann_{S^{-1}R}\, S^{-1}M = ann_{S^{-1}R}\, r/t\, S^{-1}M$ $\forall \frac{r}{t} \notin ann_{S^{-1}R}\, S^{-1}M$, since $r/t\, S^{-1}M \subseteq S^{-1}M$

so $ann_{S^{-1}R}\, S^{-1}M \subseteq ann_{S^{-1}R}\, r/t\, S^{-1}M$ …(1)

To prove $ann_{S^{-1}R}\, r/t\, S^{-1}M \subseteq ann_{S^{-1}R}\, S^{-1}M$

Let $y/t' \in ann_{S^{-1}R}\, r/t\, S^{-1}M$ so $y/t' \cdot r/t\, S^{-1}M = 0$ which implies that $yr/tt' S^{-1}M = 0$ where $yr \in M$, $tt' \in S$ so $yr/tt' S^{-1}M = 0$ which implies that $yr/tt' M/S = 0$ so $yrM = 0$. Hence $y \in ann_R rM = ann_R M$.

Since $y \in ann_R M$ so $yM = 0$. Thus $yM/ts = 0$ so $y/t \cdot S^{-1}M = 0$, $y/t \cdot \in ann_R S^{-1}M$, hence $ann_{S^{-1}R}\, r/t\, S^{-1}M \subseteq ann_{S^{-1}R}\, S^{-1}M$ …(2)

From (1) and (2) we have $ann_{S^{-1}R}\, S^{-1}M = ann_{S^{-1}R}\, r/t\, S^{-1}M$, so $S^{-1}M$ is W.q.p module.

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حول الموديولات الشبه الأوليه الضعيفة

منتهى عبد الرزاق حسن
قسم الرياضيات، كلية التربية الاساسية، الجامعة المستنصرية

استلم البحث في: 3 نيسان 2011 قبل البحث في: 18 تشرين الاول 2011

الخلاصة

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الكلمات المفتاحية: الموديول الأولي ، الموديول الشبه الأولي ، الموديول الشبه الأولي الضعيف.