# Approximations of Entire Functions in Locally Global Norms

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## Abstract

The purpose of this paper is to evaluate the error of the approximation of an entire function by some discrete operators in locally global quasi-norms ($L_{\delta,p}$-space), we intend to establish new theorems concerning that Jackson polynomial and Valee-Poussin operator remain within the same bounds as bounded and periodic entire function in locally global norms ($L_{\delta,p}$), $(0 < p \le 1)$.
**Key words :** Entire functions, bounded masurable functions, quasi-normed space.

## Introduction and Preliminaries

Al-Abdulla, A. [1], Al-Saidy, S.K. [2] and E.S.Bhayah [3] gave estimation for approximation of bounded measurable functions with some discrete operators in $L_p$-norm $(0 < p \le 1)$.

Here, we give an estimation for approximation of entire functions in $L_{\delta,p}$-space.

Let $X = [-\pi,\pi]$ we denote the set of all $2\pi$-periodic bounded measurable function with usual sup-norm by $L_\infty$, such that

$L_\infty(X) = \{f : f \text{ is } 2\pi\text{-periodic bounded measurable function}\}$ with norm

$$\|f\|_\infty = \sup\{|f(x)| \forall x \in X\} < \infty \qquad \ldots(1.1)$$

and the $L_p$-norm $(1 \le p < \infty)$ of $f \in L_p$ by $\|f\|_{L_p}$, such that

$$L_p(X) = \left\{ f : \|f\|_p = \left(\int_X |f(x)|^p dx\right)^{\frac{1}{p}} < \infty \right\}; \|f\|_{L_p(X)} = \|f\|_p \qquad \ldots(1.2)$$

Now let us consider the Dirich let kernel of degree n, [4]

$$D_n(u) = \frac{1}{2} + \sum_{v=1}^{n} \cos(vu) \ni u \in R,\ n=0,1,\ldots \qquad \ldots(1.3)$$

Let

$$K_n(u) = \frac{1}{n+1}[D_0(u) + D_1(u) + \ldots + D_n(u)] \qquad \ldots(1.4)$$

be the Fejer kernel of degree not grater than n.

$$J_n(f,x) = \frac{2}{n+1}\sum_{k=0}^{n} f(x_{k,n})K_n(X - X_{k,n}) \qquad \ldots(1.5)$$

where $X_{k,n} = \dfrac{2K\pi}{n+1}, (K = 0,1,2,\ldots,n)$, be the so called Jackson polynomial of function $f \in L_\infty$.

$$V_{2n}(t)=\frac{1}{n+1}[D_n(t)+D_{n+1}(t)+D_{2n}(t)] \quad \ldots(1.6)$$

Let $X_j=\dfrac{2\pi j}{3n+1}$, j=0,1,…,3n. Then we define the following operator.

$$V_{2n,3n}(f,X)=\frac{2}{3n+1}\sum_{j=0}^{3n} f(X_j)V_{2n}(X-X_j) \quad \ldots(1.7)$$

be the valee-poussin discrete operator of $2\pi$-periodic bounded measurable function.

The unique linear trigonometric polynomial which is interpolating a given function $f \in L_p(X)$ at the point $X_j$ is denote by $I_n(t)$ which has the representation:

$$I_n(f,X)=\frac{2}{2n+1}\sum_{j=0}^{2n} f(X_j)D_n(X-X_j) \quad \ldots(1.8)$$

Now let $B_n$ be the set of all entire functions, since the derivative of polynomial exists every where, then we get that every polynomial is an entire function [5], so we consider that $f \in B_n$ and $J_n(f) \in B_n$, $V_{2n,3n}(f) \in B_n$.

Let n, k be positive integers, $(0 < p \le 1)$ and $(\delta > 0)$ are fixed numbers which will be used for the degree of approximating polynomial, for the rate order of modulus and for the space $L_{\delta,p}$ respectively.

We consider the locally global norm for $(\delta > 0)$, $(0 < p \le \infty)$

$$\|f\|_{\delta,p}=\left(\int_X \sup\left\{|f(y)|, y\in\left[x-\frac{\delta}{2}, x+\frac{\delta}{2}\right]\right\}^p dx\right)^{\frac{1}{p}}, X\in[-\pi,\pi]. \quad \ldots(1.9)$$

Now the $k^{th}$ average modulus of smoothness for $f \in L_{\delta,p}$ are defined by the following respectively, [6], [7]

$$\left\{\begin{array}{l} \tau_k(f,\frac{1}{n})_p=\left\|W_k(f,.,\frac{1}{n})\right\|_p, \\ \tau_k(f,\frac{1}{n})_{\delta,p}=\left\|W_k(f,.,\frac{1}{n})\right\|_{\delta,p} \end{array}\right\} \quad \ldots(1.10)$$

where the $k^{th}$ modulus of smoothness for $f \in L_{\delta,p}$, $k \in \mathbb{N}$ is defined by

$$W_k(f,x,\frac{1}{n})=\sup\left\{\left|\Delta_h^k f(t)\right| : t, t+kh\in\left[x-\frac{k\delta}{2}, x+\frac{k\delta}{2}\right]\cap X\right\} \quad \ldots(1.11)$$

Now, we set $\Delta_h^k f(t)=\left\{\begin{array}{ll}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}f(t+mh) & \text{if } t \text{ or } t+kh\in X \\ 0 & \text{otherwise}\end{array}\right\}.$

In the following we recall some theorems which are needed:-

**Theorem 1.1:** [6]

If $f \in B_n$, then for $(0 < p \le 1)$ and $(\delta > 0)$, we have,

$$\|f\|_{\delta,p} \le c(p)[(1+n\delta)^{1-p}(ns)]^{\frac{1}{p}}\|f\|_p .$$

**Theorem 1.2:** [3]

If f ∈ 2π-periodic bounded measurable functions, then for (0 < p ≤ 1)

$$\|f - J_n(f)\|_p \le C(p)\tau_1(f, \frac{-1}{n})_p .$$

**Theorem 1.3:** [3]

If f ∈ 2π-periodic bounded measurable function, then for (0 < p ≤ 1)

$\|f - V_{2n,3n}(f)\|_p \le C(p,k,\ell)\tau_k(f, \frac{1}{2n})_p$, where n=1,2,… and (p,k,ℓ) is a constant depends on p, k and ℓ.

**Theorem 1.4:** [3]

Let f be 2π-periodic bounded measurable function, then for (0 < p ≤ 1), we have

$\|f - I_n(f)\|_p \le C(p,k,\ell)\tau_k(f, \frac{1}{n})_p$, where p,k,ℓ is a constant depends on p, k and ℓ.

# Main Results

We shall prove direct inequality to find the degree of approximation of 2π-periodic entire function by some discrete operators in ($L_{\delta,p}$) spaces, (0 < p ≤ 1).

**Lemma 2.1:**

Let f be 2π-periodic entire function, then for (0 < p ≤ 1), we have

$$\tau_k(f, \frac{1}{n})_p \le \tau_k(f, \frac{1}{n})_{\delta,p} .$$

**Proof:**

$$\tau_k(f, \frac{1}{n})_p = \left\| W_k(f,.,\frac{1}{n}) \right\|_p$$

$$= \left\| \sup\left\{ |\Delta_h^k f(t)|; t, t+kh \in \left[ x - \frac{k}{2n}, x + \frac{k}{2n} \right] \cap X \right\} \right\|_p$$

$$= \left\| \sup\left\{ \left| \sum_{i=0}^{k} (-1)^{i+k} \binom{k}{i} f(t+ih) \right|; t, t+kh \in \left[ x - \frac{k}{2n}, x + \frac{k}{2n} \right] \cap X \right\} \right\|_p$$

$$= \left( \int_X \left| \sup\left\{ \left| \sum_{i=0}^{k} (-1)^{i+k} \binom{k}{i} f(t+ih) \right| \right|^p ; t, t+kh \in \left[ x - \frac{k}{2n}, x + \frac{k}{2n} \right] \cap X \right\} dx \right)^{\frac{1}{p}}$$

$$\le \left( \int_X \sup \left| \sup\left\{ \left| \sum_{i=0}^{k} (-1)^{i+k} \binom{k}{i} f(t+ih) \right| \right|^p ; t, t+kh \in \left[ y - \frac{k}{2n}, y + \frac{k}{2n} \right] \cap X \right\} y \in \left[ x - \frac{k}{2n}, x + \frac{k}{2n} \right] \Bigg|^p dx \right)^{\frac{1}{p}}$$

$$= \left\| \sup\left\{ |\Delta_h^k f(t)|; t, t+kh \in \left[ x - \frac{k}{2n}, x + \frac{k}{2n} \right] \cap X \right\} \right\|_{\delta,p}$$

$$= \left\| W_k(f,.,\frac{1}{n}) \right\|_{\delta,p}$$

$$= \tau_k(f, \frac{1}{n})_{\delta,p}$$

**Theorem 2.2:**

Let f be 2π-periodic bounded measurable entire function, (f ∈ $L_{\delta,p}$), (0 < p ≤ 1), we have

$\|f - J_n(f)\|_{\delta,p} \le C(p)\tau_1(f,\frac{1}{n})_{\delta,p}$, where C(p) is a constant depends only on p.

**Proof:**

By theorem (1.1), we get

$$\|f - J_n(f)\|_{\delta,p} \le C(p)[1 + (1\_n\delta)^{1-p}(n\delta)^p]^{\frac{1}{p}}\|f - J_n(f)\|_p .$$

Now since ($\delta > 0$), then

$$\|f - J_n(f)\|_{\delta,p} \le C_1(p)\|f - J_n(f)\|_p .$$

Then by using theorem (1.2) and lemma (2.1), we get that

$$\|f - J_n(f)\|_{\delta,p} \le C_2(p)\,\tau_1(f,\frac{1}{n})_p$$

$$\le C(p)\tau_1(f,\frac{1}{n})_{\delta,p}$$

**Theorem 2.3:**

Let f be $2\pi$-periodic bounded measurable entire function, ($f \in L_{\delta,p}$), ($0 < p \le 1$), we have $\|f - V_{2n,3n}(f)\|_{\delta,p} \le C(p,k,\ell)\,\tau_k(f,\frac{1}{2n})_{\delta,p}$ ,where p,k,ℓ is a constant depends on p, k and ℓ.

**Proof:**

By using theorem (1.1), we get

$$\|f - V_{2n,3n}(f)\|_{\delta,p} \le C_1(p)[1 + (1 + n\delta)^{1-p}(n\delta)^p]^{\frac{1}{p}}\|f - V_{2n,3n}(f)\|_p .$$

Since $\delta = \frac{1}{n}$, then

$$\|f - V_{2n,3n}(f)\|_{\delta,p} \le C_2(p)\|f - V_{2n,3n}(f)\|_p .$$

Now by using theorem (1.3) and lemma (2.1), we have

$$\|f - V_{2n,3n}(f)\|_{\delta,p} \le C(p,k,\ell)\,\tau_k(f,\frac{1}{2n})_p$$

$$\le C(p,k,\ell)\tau_k(f,\frac{1}{2n})_{\delta,p} .$$

**Theorem 2.4:**

Let f be $2\pi$-periodic bounded measurable entire function, ($f \in L_{\delta,p}$), ($0 < p \le 1$), we have $\|f - I_n(f)\|_{\delta,p} \le C(p,k,\ell)\,\tau_k(f,\frac{1}{2n})_{\delta,p}$, where p,k,ℓ is a constant depends on p, k and ℓ.

**Proof:**

By using theorem (1.1), we get

$$\|f - I_n(f)\|_{\delta,p} \le C_1(p)[1 + (1 + n\delta)^{1-p}(n\delta)^p y_p]\|f - I_n(f)\|_p .$$

Since $\delta = \frac{1}{n}$, then

$$\|f - I_n(f)\|_{\delta,p} \le C_2(p)\|f - I_n(f)\|_p .$$

Then by using theorem (1.4) and lemma (2.1), we get

$$\|f - I_n(f)\|_{\delta,p} \le C(p,k,\ell)\,\tau_k(f,\frac{1}{n})_p$$

$$\le C(p,k,\ell)\tau_k(f,\frac{1}{n})_{\delta,p}.$$

## Conclusion

We found the degree of approximation of entire functions by using Jackson, Vallee Pouson and interpolation polynomials in locally quasi-norms $L_{\delta,p}$ ( $0 < p < 1$).

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# تقريب الدوال الداخلية بواساطة المتعددات المتقطعة في الفضاءات المحلية

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**الخلاصة**

الغرض من هذا البحث هو حساب مقدار الخطأ لتقريب الدوال الداخلية بواساطة بعض المؤثرات المتقطعة في الفضاءات شبه المحلية باستعمال الوسيط $\tau_k(\delta, \frac{1}{n})_p$ .

**الكلمات المفتاحية :** الدوال الداخلية ، الدوال محدودة القياس ، الفضاء شبه المعياري.