المقاسات الجزئية الأولية المضادة

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الخلاصة

لتكن R حلقة ابدالية ذو محايد وليكن M مقاساً احادياً على R. ليكن N مقاس جزئي فعلي من M. يقال عن N مقاساً جزئياً اولي مضاد اذا كان المقاس $\frac{M}{N}$ اولي مضاد، حيث ان المقاس $\frac{M}{N}$ يسمى اولي مضاد اذا كان لكل $r \in R$ ، اما $r\frac{M}{N} = O_{\frac{M}{N}}$ أو $r\frac{M}{N} = \frac{M}{N}$.

في هذا البحث درسنا المقاسات الجزئية الأولية المضادة واعطينا العديد من الخواص المتعلقة بهذا المفهوم.

الكلمات المفتاحية: المقاسات الجزئية الاولية المضادة– المقاسات الجزئية الثانية المقاسات الثانية (المضادة الاولية)– المقاسات الثانوية.

Coprime Submodules

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Abstract

Let R be a commutative ring with unity and let M be a unitary R-module. Let N be a proper submodule of M, N is called a coprime submodule if $\frac{M}{N}$ is a coprime R-module, where $\frac{M}{N}$ is a coprime R-module if for any $r \in$ R, either $r\frac{M}{N} = 0_{\frac{M}{N}}$ or $r\frac{M}{N} = \frac{M}{N}$.

In this paper we study coprime submodules and give many properties related with this concept.

Key words: Coprime submodules, second submodule, second (coprime) module, secondary module.

Introduction

Let R be a commutative ring with unity and let M be a unitary R-module. It is well-known that a proper submodule N of an R-module M is called prime if whenever $r \in$R, $x \in$M, $rx \in$N implies $x \in$N or $r \in$ [N:M], where [N:M]={$r \in$R: rM$\subseteq$N}. M is called a prime module if $\underset{R}{\text{ann}}\,M = \underset{R}{\text{ann}}\,N$ for all nonzero submodule N of M, equivalently M is a prime module iff (0) is a prime submodule.

S.Yassem in [7], introduced the notions of second submodules and second modules, where a submodule N of M is called second if for any $r \in$R, the homothety $r^* \in$End M, is either zero or surjective, where $r^*(m) = r\,m$, $\forall\ m \in$ M. It follows that N is a second submodule iff for each $r \in$R, either rN = 0 or rN = N. M is called a second module if M is a second submodule of itself.

For an R-module M, the following statements are equivalent:

(1) M is a second module.
(2) For each $r \in$R, either rM = 0 or rM = M.
(3) ann M = ann $\frac{M}{N}$ for all proper submodules N of M.
(4) ann M = ann $\frac{M}{N}$ for all fully invariant sub3
(5) modules N of M.
(6) ann M = W(M), where W(M)={$r \in$R:$r^* \in$End M, r^* is not surjective}.

Notice (1) ⇔ (2) is clear, (1) ⇔ (5) [7,lemma 1.2], (1) ⇔ (3) [3, theorem 2.1.6], (3) ⇔ (4) [6, theorem 1.3.2].

Notice that statement (3) and statement (4) are used to define coprime module by S. Annin in [2] and I.E Wijayart in [6], respectively.

Moreover Rasha in [3] studied coprime modules and give some generalizations of these modules, (see [3]).

J.Abuhilail in [1], introduced the notion of coprime submodule, where a proper submodule N of M is called coprime if ann $\frac{M}{N}$ = W($\frac{M}{N}$); that is N is a coprime submodule if $\frac{M}{N}$ is a coprime R-module.

Our aim in this paper is to study coprime submodules, we give the basic properties about this concept. Also, we study coprime submodules in certain classes of modules.

1- Coprime Submodules

We give the basic properties related with coprime submodules. Also, we study their behaviour in certain classes of modules.

Following J.Abuhilail in [1], a proper submodule N of an R-module M is called coprime if $\frac{M}{N}$ is a coprime R-module.

An ideal I of a ring R is called coprime ideal iff $\frac{R}{I}$ is a coprime R-module.

1.1 Remarks and Examples:

(1) N is coprime submodule iff for each $r \in$ R either $r\frac{M}{N} = 0_{\frac{M}{N}}$ = N or $r\frac{M}{N} = \frac{M}{N}$, that is N is a coprime submodule if for each $r \in$R, either $r \in$ [N:M] or for any $m \in$ M, there exists $m' \in$ M such that $m - r\,m' \in$ N.

(2) Z is a coprime submodule of the Z-module Q, since $\frac{Q}{Z}$ is a coprime Z-module [4], [6]. Note that Z is not coprime Z-module, since when $r = 2 \neq 0$, 2Z ≠ Z.

(3) Every submodule N of the Z-module Z_{p^∞} is a coprime submodule, since Z_{p^∞}/N ≅ Z_{p^∞} and Z_{p^∞} is a coprime Z-module, hence Z_{p^∞}/N is a coprime Z-module.

(4) Let M be a coprime R-module, then every proper submodule N of M is a coprime submodule.

proof: Since M is a coprime R-module, then by [3,cor. 2.1.12], $\frac{M}{N}$ is a coprime R-module, for all N $\underset{\neq}{<}$ M. Hence N is a coprime submodule.

(5) If N is a maximal submodule of an R-module M, then N is a coprime submodule.

proof: Since N is maximal, $\frac{M}{N}$ is a simple R-module, hence $\frac{M}{N}$ is a coprime R-module. Thus N is a coprime submodule.

(6) The converse of (4) is not true in general for example, Z is a coprime submodule of the Z-module Q (see 1.1 (2)) but Z is not a maximal submodule of Q.

(7) Let M be an R-module, let I be an ideal of R such that I ⊆ ann M, let N $\underset{\neq}{<}$ M. Then N is a coprime R-submodule of M ⇔ N is a coprime $\overline{R}$ -submodule of M, where $\overline{R}$ =R / I.

proof: ($\Rightarrow$) Let N be a coprime R-submodule. Then $\frac{M}{N}$ is a coprime R-module and hence by [3, cor. 2.1.9], $\frac{M}{N}$ is coprime $\overline{R}$-module. Thus N is a coprime $\overline{R}$-module.

($\Leftarrow$) The proof is similarly.

1.2 Proposition:

If N is a coprime submodule, then [N:M] is a prime ideal.

proof: Since N is a coprime submodule, $\frac{M}{N}$ is coprime R-module. Hence ann$\frac{M}{N}$ is a prime ideal of R [3, note 2.1]. But ann$\frac{M}{N}$ = [N:M], so [N:M] is a prime ideal.

Recall that an R-module M is called secondary if for each $r \in$ R, either $r\,m = 0$ or r^nM = M, for some $n \in Z_+$. [7].

We have the following:

1.3 Proposition:

Let M be a secondary R-module, let $N \underset{\neq}{<} M$. Then N is a coprime submodule iff [N:M] is a prime ideal of R.

proof: ($\Rightarrow$) It follows by prop. 1.2.

($\Leftarrow$) Since M is a secondary R-module, then $\frac{M}{N}$ is a secondary R-module. But [N:M] = ann$\frac{M}{N}$ is a prime ideal, so by [3,prop.1.2.6], $\frac{M}{N}$ is a coprime R-module, hence N is a coprime submodule.

1.4 Proposition:

Let N be a proper submodule of an R-module M. Then N is a coprime submodule iff [N:M]=[W:M] for all W $\supset$ N.

proof: If N is a coprime submodule, then $\frac{M}{N}$ is a coprime R-module. Hence ann $\frac{M}{N}$= ann $\frac{M/N}{W/N}$ for all W$\supset$N. It follows that ann$\frac{M}{N}$=ann$\frac{M}{W}$; that is [N:M]= [W:M].

If [N:M] = [W:M], for all W $\supset$ N, then ann$\frac{M}{N}$=ann$\frac{M}{W}$. But $\frac{M}{W} \cong \frac{M/N}{W/N}$, so ann $\frac{M}{N}$=ann$\frac{M/N}{W/N}$ and $\frac{M}{N}$ is a coprime R-module. Thus N is a coprime submodule.

1.5 Proposition:

Let W be a coprime submodule of M and let $N \underset{\neq}{<} M$ such that N $\supset$ W. Then N is a coprime submodule of M and $\frac{N}{W}$ is a coprime submodule of $\frac{M}{W}$.

proof: Since W is a coprime submodule, then $\frac{M}{W}$ is a coprime R-module. Hence by [Rem and Ex. 1.1 (4)], $\frac{N}{W}$ is a coprime submodule of $\frac{M}{W}$. Also $\frac{M}{W}$ is a coprime R-module implies (M/W) / (N/W) is a coprime R-module [3,cor. 2.1.12]. But (M/W) / (N/W) ≅ M / N, hence M / N is a coprime module by [3, Cor. 2.1.14]. Thus N is a coprime submodule of M.

1.6 Proposition:

Let M be an R-module, let N, W be proper submodules of M, N ⊇ W such that $\frac{N}{W}$ is a coprime submodule of $\frac{M}{W}$.Then N is a coprime submodule of M.

proof: Since $\frac{N}{W}$ is a coprime submodule of $\frac{M}{W}$, we have (M/W) / (N/W) is a coprime module. Thus M / N is a coprime module and so N is a coprime submodule of M.

The following results follow directly by proposition 1.5.

1.7 Corollary:

If N is a coprime submodule of an R-module M, I an ideal of R. Then $[N \underset{M}{:} I]$ is a coprime submodule of M.

1.8 Corollary:

Let A, B be proper submodules of an R-module M. If A or B is a coprime submodule and A + B ≠ M. Then A + B is a coprime submodule of M.

1.9 Proposition:

Let I be a proper ideal of a ring R. Then I is a coprime ideal iff I is a maximal ideal of R.

proof: If I is a coprime ideal of R, then R/I is a coprime R-module. But R/I is a multiplication R-module, so by [3,Rem. And Ex. 2.1.3(5)] R/I is simple R-module. Thus I is a maximal ideal of R.

The converse follows by (Rem. And Ex. 1.1.(5)).

1.10 Corollary:

Let R be a ring. The following are equivalent:

(1) (0) is a coprime submodule of R.

(2) R/(0) ⊔ R is a coprime ring (that is R is a field).

(3) (0) is a maximal ideal of R.

1.11 Corollary:

Let R be a PID, let I be a nonzero proper ideal of R. Then the following are equivalent:

(1) I is a coprime ideal of R.

(2) I is a maximal ideal of R.

(3) I is a prime ideal of R.

1.12 Note:

If N is a coprime submodule of an R-module M. Then it is not necessary that [N:M] is a coprime ideal of R, as the following example shows:

Z is a coprime submodule of the Z-module Q but [Z:Q] = (0) is not a maximal ideal of Z, that is (0) is not coprime ideal of Z.

1.13 Proposition:

Let M be a multiplication R-module, let N be a proper submodule of M. Then N is a coprime submodule iff [N:M] is a coprime ideal of R.

proof: If N is a coprime submodule of M, then $\frac{M}{N}$ is a coprime R-module. But M is a multiplication R-module implies $\frac{M}{N}$ is a multiplication R-module. Hence by [3,Rem. and Ex. 2.1.3(5)] $\frac{M}{N}$ is a simple R-module. Thus N is a maximal submodule of M which implies that [N:M] is a maximal ideal. Then by prop. 1.9, [N;M] is a coprime ideal.

Conversely, if [N:M] is a coprime ideal of R, then by prop. 1.9, [N:M] is a maximal ideal of R. Now M is a multiplication module and [N;M] is a maximal ideal of R implies that N=[N;M]M is a maximal submodule of M. Thus by Rem. and Ex. 1.1 (5), N is a coprime submodule of M.

1.14 Corollary:

Let M be a multiplication R-module and let $N < M$. The following are equivalent:

(1) N is a coprime submodule of M.
(2) [N:M] is a coprime ideal of R.
(3) [N:M] is a maximal ideal of R.
(4) N is a maximal submodule of M.

proof: (1) $\Leftrightarrow$ (2) it follows by prop. 1.13.
(2) $\Leftrightarrow$ (3) it follows by prop. 1.9.
(4) $\Rightarrow$ (1) by Rem. and Ex. 1.1 (5).
(3) $\Rightarrow$ (4) Since M is multiplication, and [N:M] is a maximal ideal, then N is a maximal submodule of M.

The following result shows that a homomorphic image of a coprime submodule is a coprime submodule.

1.15 Theorem:

Let $\psi: M \longrightarrow M'$ be an R-epimorphism, let $N < M$. If N is a coprime submodule of M, then $\psi(N)$ is a coprime submodule of M'.

proof: To prove $\psi(N)$ is a coprime submodule of M', we must prove $\frac{M'}{\psi(N)}$ is a coprime R-module, so we must show that $r\frac{M'}{\psi(N)} = \frac{M'}{\psi(N)}$ for all $r \notin \text{ann}\frac{M'}{\psi(N)}$. First $r \notin \text{ann}\frac{M'}{\psi(N)}$, means that $r \notin [\psi(N):M']$. It is easy to check that $[N:M] \subseteq [\psi(N):M']$. Hence $r \notin [N:M] = \text{ann}\frac{M}{N}$. On the other hand N is a coprime submodule, implies $\frac{M}{N}$ is a coprime R-module. Hence $r\frac{M}{N} = \frac{M}{N}$ since $r \notin \text{ann}\frac{M}{N} = [N:M]$. Now, let $y + \psi(N) \in \frac{M'}{\psi(N)}$, so $y = \psi(m)$ for some $m \in N$, since ψ is an epimorphism. Thus $y + \psi(N) = \psi(m) + \psi(N) = \psi(m + N)$. Hence there exists $m' \in M$ such that. $m + N = r\,m + N$, so $y + \psi(N) = \psi(r\,m' + N) = r\,\psi(m') + N = r\,(\psi(m') + N) \in r\frac{M'}{N}$. Thus $r\frac{M'}{\psi(N)} = \frac{M'}{\psi(N)}$ and so $\frac{M'}{\psi(N)}$ is a coprime R-module. Hence $\psi(N)$ is a coprime submodule of M'.

Now, we turn our attention to direct sum of coprime submodules.

1.16 Theorem:

Let M_1, M_2 be R-modules, let $N_1 < M_1$, $N_2 < M_2$ such that $\operatorname{ann}\frac{M_1}{N_1} = \operatorname{ann}\frac{M_2}{N_2}$. Then $N = N_1 \oplus N_2$ is a coprime submodule of M iff N_1 is a coprime submodule of M_1, N_2 is a coprime submodule of M_2.

proof: ($\Rightarrow$) Let $p_1:M_1\oplus M_2 \longrightarrow M_1$, $p_2:M_1\oplus M_2 \longrightarrow M_2$ be the natural projection. Hence $p_1(N_1\oplus N_2) = N_1$, $p_2(N_1\oplus N_2) = N_2$ and so by theorem 1.15, N_1 is a coprime submodule of M_1, N_2 is a coprime submodule of M_2.

Conversely, to prove $N_1\oplus N_2$ is a coprime submodule of $M_1\oplus M_2$. Since N_1, N_2 are coprime submodules of M_1, M_2 respectively, then $\frac{M_1}{N_1}$ and $\frac{M_2}{N_2}$ are coprime R-module and since $\operatorname{ann}\frac{M_1}{N_1} = \operatorname{ann}\frac{M_2}{N_2}$ it follows that $\frac{M_1}{N_1} \oplus \frac{M_2}{N_2}$ is a coprime R-module (see [7], [3,prop. 2.3.3). But it is easy to check that $\frac{M_1 \oplus M_2}{N_1 \oplus N_2} \cong \frac{M_1}{N_1} \oplus \frac{M_2}{N_2}$. Hence by [3,cor. 2.1.14], $\frac{M_1 \oplus M_2}{N_1 \oplus N_2}$ is a coprime R-module. Thus $N_1\oplus N_2$ is a coprime submodule of $M_1\oplus M_2$.

1.17 Remark:

The condition $\operatorname{ann}\frac{M_1}{N_1} = \operatorname{ann}\frac{M_2}{N_2}$ is necessary condition in Th. 14, as the following example shows:

Consider the Z-module Z. Let $N_1 = 2Z$, $N_2 = 3Z$, N_1, N_2 are maximal submodules of Z, so N_1, N_2 are coprime submodules of Z (see Rem. 1.1(5)). Let $N = N_1 \oplus N_2 = 2Z \oplus 3Z < Z\oplus Z$. It is clear that $\operatorname{ann}\frac{Z}{N_1} \neq \operatorname{ann}\frac{Z}{N_2}$. Now $\frac{Z \oplus Z}{N_1 \oplus N_2} \cong \frac{Z}{N_1} \oplus \frac{Z}{N_2} \cong Z_2 \oplus Z_3 \cong Z_6$. But Z_6 is not a coprime Z-module, so $\frac{Z \oplus Z}{N_1 \oplus N_2}$ is not a coprime Z-module. Thus $N_1 \oplus N_2$ is not a coprime submodule of $Z\oplus Z$.

The following property explains the behaviour of coprime submodules under localization.

1.18 Proposition:

Let S be a multiplicative subset of a ring R. Let N be a proper submodule of an R-module M such that $S^{-1}N \neq S^{-1}M$. If N is a coprime submodule of M, then $S^{-1}N$ is coprime sbmodule of $S^{-1}M$.

proof: N is a coprime submodule of M implies $\frac{M}{N}$ is a coprime R-module, then by [3,prop.2.1.38], $S^{-1}\left(\frac{M}{N}\right)$ is a coprime $S^{-1}R$-module. But [5,lemma 9.12,p.173],

$S^{-1}\left(\frac{M}{N}\right) \cong \frac{S^{-1}M}{S^{-1}N}$, so $\frac{S^{-1}M}{S^{-1}N}$ is a coprime $S^{-1}R$-module. Hence $S^{-1}N$ is a coprime submodule of $S^{-1}M$.

Recall that an R-module M is antihopfian if M = M/N for all $N \underset{\neq}{<} M$ (4).

Hence we get the following result directly.

1.19 Remark:

Let M be an antihopfian R-module. Then every submodule of M is coprime submodule.

proof: Since $M \cong \frac{M}{N}$, ann M = ann $\frac{M}{N}$, that is M is coprime R-module. Then by (Rem. and Ex. 1.1(4)) every proper submodule is coprime submodule.

1.20 Proposition:

Let M be a finitely generated R-module, let $N \underset{\neq}{<} M$. If N is a coprime submodule, then N is prime.

proof: Since N is a coprime submodule, M/N is a coprime R-module. But M is a finitely generated R-module, so M / N is finitely generated. Hence by [3,Th. 2.4.8], M/N is a prime R-module and hence $O_{M/N}$ = N is a prime submodule of $\frac{M}{N}$. It follows that N is a prime submodule of M.

1.21 Remark:

The condition M is finitely generated in prop. 2.1 is necessary condition, as the following example shows.

Z is a coprime submodule of the Z-module Q and Q is not finitely generated. Also Z is not a prime submodule of Q.

1.22 Corollary:

Let M be a Noetherian coprime R-module, then every proper submodule of M is prime.

proof: It follows directly by prop. 1.20.

1.23 Proposition:

Let M be an R-module such that $rM \cap N = rN$ for all $r \in R$ and for all $N \underset{\neq}{<} M$. Then every prime submodule is a coprime submodule.

proof: Let N be a prime submodule of M. Let $W \supset N$. We shall prove that: $r\frac{M}{N} \cap \frac{W}{N} = r\frac{W}{N}$ as follows: let $x \in r\frac{M}{N} \cap \frac{W}{N}$, so $x = w + N = r\,(m + N)$ for some $w \in W$, $m \in M$. Hence $r\,m - w \in N \subset W$. Thus $r\,m \in W$, which implies that $rm \in rM \cap W = rW$ and hence $r\,m = r\,y$ for some $y \in W$. Then $rm + N = r\,y + N$, that is $r\,(m + N) = r\,(y + N) \in r\frac{W}{N}$. Thus $r\frac{M}{N} \cap \frac{W}{N} = r\frac{W}{N}$. On the other hand, N is a prime submodule of M implies $\frac{M}{N}$ is a prime R-module. Then by [3,prop. 2.4.1,p.54] $\frac{M}{N}$ is a coprime R-module and hence N is a coprime submodule.

1.24 Corollary:

Let R be a regular ring (in sence of Von Neumann), let M be an R-module. Then every prime submodule of M is a coprime submodule of M.

proof: Since R is a regular ring, implies $rM \cap N = rN$ for all $r \in R$ and for all N<M, then the result is obtained by prop.1.23.

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