Some Results on The Complete Arcs in Three Dimensional Projective Space Over Galois Field

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Abstract

The aim of this paper is to introduce the definition of projective 3-space over Galois field GF(q), q = p^m, for some prime number p and some integer m.

Also the definitions of (k,n)-arcs, complete arcs, n-secants, the index of the point and the projectively equivalent arcs are given.

Moreover some theorems about these notations are proved.

Keywords: arcs, index, plane.

Introduction: [1]

A projective 3 – space PG(3,K) over a field K is a 3 – dimensional projective space which consists of points, lines and planes with the incidence relation between them.

The projective 3 – space satisfies the following axioms:

A. Any two distinct points are contained in a unique line.

B. Any three distinct non-collinear points, also any line and point not on the line are contained in a unique plane.

C. Any two distinct coplanar lines intersect in a unique point.

D. Any line not on a given plane intersects the plane in a unique point.

E. Any two distinct planes intersect in a unique line.

A projective space PG(3,q) over Galois field GF(q), q = p^m, for some prime number p and some integer m, is a 3 – dimensional projective space.

Now, some theorems on PG(3,q) proved in [1] and [2] are given in the following.

Theorem 1:
Every line in PG(3,q) contains exactly q + 1 points.

Theorem 2:
Every point in PG(3,q) is on exactly q + 1 lines.

Theorem 3:
Every plane in PG(3,q) contains exactly $q^2 + q + 1$ points.

Theorem 4:
Every plane in PG(3,q) contains exactly $q^2 + q + 1$ lines.

Theorem 5:

Every point in PG(3,q) is on exactly $q^2 + q + 1$ planes.

Theorem 6:

There exist $q^3 + q^2 + q + 1$ points in PG(3,q).

Theorem 7:

There exist $q^3 + q^2 + q + 1$ planes in PG(3,q).

Theorem 8:

Any line in PG(3,q) is on exactly q +1 planes.

Definition 1: [1]

A (k,n) – arc A in PG(3,q) is a set of k points such that at most n points of which lie in any plane, $n \geq 3$. n is called the degree of the (k,n) – arc.

Definition 2:

In PG(3,q), if A is any (k,n) – arc, then an (m-secant) of A is a plane ℓ such that $|\ell \cap A| = m$.

Definition 3: [1,2]

A point N not on a (k,n)-arc A has index i if there exists exactly i (n –secants) of A through N, one can denote the number of points N of index i by C_i.

Definition 4:

(k,n)-arc A is complete if it is not contained in any $(k + 1,n)$-arc.

From definitions 3 and 4, it is concluded that the (k,n)-arc is complete iff $C_0 = 0$. Thus the (k,n)-arc is complete iff every point of PG(3,q) lies on some n-secant of the (k,n)-arc.

Definition 5: [1,3]

Let T_i be the total number of the i – secants of a (k,n) – arc A, then the type of A denoted by $(T_n, T_{n-1}, \ldots, T_0)$.

Definition 6: [1]

Let (k_1,n) – arc A is of type $(T_n, T_{n-1},\ldots, T_0)$ and (k_2,n) – arc B is of type $(S_n,S_{n-1},\ldots,S_0)$, then A and B have the same type iff $T_i = S_i$, for all i, in this case they are projectively equivalent.

Theorem 9:

Let t(P) represents the number of 1-secants (planes) through a point P of a (k,n) – arc A and let T_i represent the numbers of i – secants (planes) for the arc A in PG(3,q), then:

1. $t = t(P) = q^2 + q + 2 - k - \dfrac{(k-1)(k-2)}{2} - \cdots - \dfrac{(k-1)(k-2)\cdots(k-(n-1))}{(n-1)!}$
2. $T_1 = k\,t$
3. $T_2 = \dfrac{k(k-1)}{2}$
4. $T_3 = \dfrac{k(k-1)(k-2)}{3!}$

5. $T_n = \dfrac{k(k-1)\cdots(k-n+1)}{n!}$

6. $T_0 = q^3 + q^2 + q + 1 - k\,t - \dfrac{k(k-1)}{2} - \dfrac{k(k-1)(k-2)}{3!} - \cdots - \dfrac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$

Proof :

1. there exist $(k-1)$ 2-secants to A through P and there exist $\binom{k-1}{2}$ (3-secants) to A through P, and so there exist $\binom{k-1}{n-1}$ n - secants to A through P, and since there exist exactly $q^2 + q + 1$ planes through P, then the number of the 1-secants through P:

$$t(P) = q^2 + q + 1 - (k-1) - \binom{k-1}{2} - \cdots - \binom{k-1}{n-1}$$

$$= q^2 + q + 2 - k - \frac{(k-1)(k-2)}{2} - \cdots - \frac{(k-1)(k-2)\cdots(k-n+1)}{(n-1)!} = t.$$

2. T_1 = the number of 1-secants to A, since each point of A has t (1-secants) and the number of the points is k, then $T_1 = k\,t$.

3. T_2 = the number of 2-secants to A, which is the number of planes passing through any two points of A. Hence $T_2 = \binom{k}{2} = \dfrac{k(k-1)}{2}$.

4. T_3 = the number of 3-secants of A, which is the number of planes passing through any three points of A. Hence $T_3 = \binom{k}{3} = \dfrac{k(k-1)(k-2)}{3!}$.

5. T_n = the number of n – secants planes to A, $T_n = \binom{k}{n} = \dfrac{k(k-1)\cdots(k-n+1)}{n!}$.

6. $q^3 + q^2 + q + 1$ represents the number of all planes, then in a (k,n) – arc of PG(3,q),
$q^3 + q^2 + q + 1 = T_0 + T_1 + T_2 + T_3 + \cdots + T_n$
$T_0 = q^3 + q^2 + q + 1 - T_1 - T_2 - T_3 - \cdots - T_n$
So
$$T_0 = q^3 + q^2 + q + 1 - k\,t - \frac{k(k-1)}{2} - \frac{k(k-1)(k-2)}{3!} - \cdots - \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}.$$

Theorem 10:

Let T_i represents the total number of the i – secants for a (k,n) – arc A in PG(3,q), then the following equations are satisfied:

1. $\sum_{i=0}^{n} T_i = q^3 + q^2 + q + 1$

2. $\sum_{i=1}^{n} i!\,T_i = k\,t + k(k-1) + k(k-1)(k-2) + \cdots + k(k-1)\cdots(k-n)$

3. $\sum_{i=2}^{n} i\,(i-1)\,T_i = k\,(k-1\,) + k\,(k-1)(k-2) + \frac{1}{2}\,k\,(k-1)(k-2)(k-3) + \cdots + \frac{1}{(n-2)!}$ $k\,(k-1)\cdots(k-n)$.

Proof :

1. $\sum_{i=0}^{n}$ T_i represents the sum of numbers of all i – secants to A, which is the number of all planes in the space. Hence $\sum_{i=0}^{n} T_i = q^3 + q^2 + q + 1$.

2. $T_1 = k\,t,\ t = q^2 + q + 2 - k - \frac{(k-1)(k-2)}{2} - \cdots - \frac{(k-1)\cdots(k-n+1)}{(n-1)!}$,

$T_2 = \frac{k\,(k-1)}{2}$, $T_3 = \frac{k\,(k-1)(k-2)}{3!}$, $T_4 = \frac{k\,(k-1)(k-2)(k-3)}{4!}$, …,

$T_n = \frac{k\,(k-1)\cdots(k-n+1)}{n\,!}$

$$\sum_{i=1}^{n} i\,!\,T_i = T_1 + 2\,!\,T_2 + 3\,!\,T_3 + \cdots + n\,!\,T_n$$

$$= k\,t + k\,(k-1\,) + k\,(k-1)(k-2) + \cdots + k\,(k-1)\cdots(k-n+1)$$

3. $\sum_{i=2}^{n} i\,(i-1)\,T_i = 2\,T_2 + 6\,T_3 + 12\,T_4 + \cdots + n\,(n-1)\,T_n$

$$= k\,(k-1\,) + k\,(k-1)(k-2) + \frac{1}{2}\,k\,(k-1)(k-2)(k-3) + \cdots + \frac{1}{(n-2)!}\,k\,(k-1)\cdots(k-n+1)$$

Theorem 11:

Let $R_i = R_i(P)$ represents the number of the i – secants (planes) through a point P of a (k,n) – arc A, in PG(3,q) then the following equations are satisfied:

1. $\sum_{i=1}^{n} R_i = q^2 + q + 1$

2. $\sum_{i=2}^{n} (i-1)!\,R_i = (k-1) + (k-1)(k-2) + \cdots + (k-1)(k-2)\cdots(k-n-1)$

$$= \sum_{i=1}^{n-1} (k-1)\cdots(k-i)$$

Proof :

1. $\sum_{i=1}^{n} R_i = R_1 + R_2 + \cdots + R_n$, $\sum_{i=1}^{n} R_i$ represents the sum of numbers of all the i – secants through a point P of the arc A, which is the number of the planes through P. Thus,

$$\sum_{i=1}^{n} R_i = q^2 + q + 1.$$

2. $\sum_{i=2}^{n} (i-1)!\, R_i = R_2 + 2!\, R_3 + 3!\, R_4 + \cdots + (n-1)!\, R_n$

From proof (1) of theorem 9, there exist $(k-1)$ 2-secants to A through P, and there exist $\binom{k-1}{2}$ 3-secants to A through P, and so there exist $\binom{k-1}{n-1}$ n-secants to A through P.

Thus $R_2 = k-1$, $R_3 = \binom{k-1}{2}$, $R_4 = \binom{k-1}{3}$, …, $R_n = \binom{k-1}{n-1}$

$R_3 = \dfrac{(k-1)!}{2!(k-3)!}$, $R_4 = \dfrac{(k-1)!}{3!(k-4)!}$, …, $R_n = \dfrac{(k-1)!}{(n-1)!(k-n)!}$

$R_3 = \dfrac{(k-1)(k-2)}{2}$, $R_4 = \dfrac{(k-1)(k-2)(k-3)}{3!}$, …, $R_n = \dfrac{(k-1)\cdots(k-(n-1))}{(n-1)!}$

$$\sum_{i=2}^{n} (i-1)!\, R_i = k-1 + \frac{2!(k-1)(k-2)}{2!} + \frac{3!(k-1)(k-2)(k-3)}{3!} + \cdots + \frac{(n-1)!(k-1)(k-2)\cdots(k-(n-1))}{(n-1)!}$$

$$= (k-1) + (k-1)(k-2) + (k-1)(k-2)(k-3) + \cdots + (k-1)(k-2)\cdots(k-(n-1))$$

$$= \sum_{i=1}^{n-1} (k-1)\cdots(k-i)$$

Theorem 12:

Let $S_i = S_i(Q)$ represent the numbers of the i – secants (planes) of a (k,n) – arc A through a point Q not in A, then the following equations are satisfied:

1. $\sum_{i=0}^{n} S_i = q^2 + q + 1$

2. $\sum_{i=1}^{n} i\, S_i = k$

Proof :

1. $\sum_{i=0}^{n} S_i$ represents the sum of the total numbers of all i – secants to A through a point Q not in A, which is equal to the number of all planes through Q. Thus $\sum_{i=0}^{n} S_i = q^2 + q + 1$.

2. $\sum_{i=1}^{n} i\, S_i = S_1 + 2\, S_2 + 3\, S_3 + \cdots + n\, S_n$

$S_1, S_2, \cdots, S_n$ represent the numbers of the i – secants of the arc A through the point Q not in A.

S_1 is the number of the 1-secants to A, each one passes through one point of A.

S_2 is the number of the 2-secants to A, each one passes through two points of A.

S_3 is the number of the 3-secants to A, each one passes through three points of A.

Also, S_n is the number of the n – secants to A, each one passes through n points of A.

Since the number of points of the (k,n) – arc A is k, then $\sum_{i=1}^{n} i\, S_i = k$.

Theorem 13:

Let C_i be the number of points of index i in S = PG(3,q) which are not on a complete (k,n) – arc A, then the constants C_i of A satisfy the following equations:

(i) $\sum_{\alpha}^{\beta} C_i = q^3 + q^2 + q + 1 - k$

(ii) $\sum_{\alpha}^{\beta} i\, C_i = \dfrac{k(k-1)\cdots(k-n+1)}{n!}\,(q^2 + q + 1 - n)$

where α is the smallest i for which $C_i \neq 0$, β be the largest i for which $C_i \neq 0$.

Proof :

The equations express in different ways the cardinality of the following sets

(i) $\{Q \mid Q \in S \setminus A\}$

(ii) $\{(Q,\pi) \mid Q \in \pi \setminus A, \pi$ an n – secant of A$\}$

for in (i), $\sum_{\alpha}^{\beta} C_i$ represents all points in the space which are not in A, then $\sum_{\alpha}^{\beta} C_i = q^3 + q^2 + q + 1 - k$, and in (ii) $\sum_{\alpha}^{\beta} i\, C_i$ represents all points in the space not in A, which are on n – secants of A, that is, each n – secant contains $q^2 + q + 1 - n$ points, and the number of the n – secants is $\binom{k}{n}$, then

$$\sum_{\alpha}^{\beta} i\, C_i = \binom{k}{n}(q^2 + q + 1 - n) = \frac{k(k-1)\cdots(k-n+1)}{n!}\,(q^2 + q + 1 - n).$$

Theorem 14:

If P is a point of a (k,n)-arc A in PG(3,q), which lies on an m-secant (plane) of A, then the planes through P contain at most $(n-1)\,q\,(q+1) + m$ points of A.

Proof :

If P in A lies on an m – secant (plane), then every other plane through P contains at most $n - 1$ points of A distinct from P. Hence the $q^2 + q + 1$ planes through P contain at most $(n-1)(q^2 + q) + m$ points of A.

References

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بعض النتائج حول الاقواس الكاملة في فضاء اسقاطي ثلاثي الابعاد حول حقل كالوا

آمال شهاب المختار
قسم الرياضيات ،كلية التربية- ابن الهيثم ،جامعة بغداد

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الخلاصة

الهدف من هذا البحث هو تقديم تعريف الفضاء الثلاثي الاسقاطي حول حقل كالوا GF(q) ، $q = p^m$، لبعض قيم p و m اذ ان p عدد أولي و m عدد صحيح.

كذلك أعطيت تعاريف الاقواس (k, n) – ، الاقواس الكاملة، القاطع – n، دليل النقطة، والاقواس المتكافئة اسقاطيا.

فضلا على ذلك برهنت بعض المبرهنات حول هذه المفاهيم.

الكلمات المفتاحية : الاقواس ، الدليل، المستوي.