N – Topological Space and Its Applications in Artificial Neural Networks

L.N.M.Tawfiq and R.N.Majeed

Department of Mathematics, College of Education -Ibn Al-Haitham, Baghdad University.

Received in March,16,2009

Accepted in March,29,2010

Abstract

In this paper we give definitions, properties and examples of the notion of type *N*-topological space. Throughout this paper *N* is a finite positive number, $N \geq 2$. The task of this paper is to study and investigate some properties of such spaces with the existence of a relation between this space and artificial Neural Networks (ANN'S), that is we applied the definition of this space in computer field and specially in parallel processing.

Introduction

Finite spaces were first studied by P.A. Alexandroff in 1937. Actually, finite spaces had been more earlier investigated by many authors under the name of simplicial complexes. There were several other contributions by Flachasmeyer in 1961, Stong in 1966 and L.Lotz in 1970, in this paper we define and study the notion of *N*- topological space and discuss some properties of finite spaces. However, the subject has never been considered as a main field of topology.

With the progress of computer technology, finite spaces have become more important. Herman in1990, Khalimsky and et. al. in 1990, kong and Kopperman in 1991 and [1] have applied them to model the computer screen.

In this paper we focus on *N*- topological space, The main importance of study is to offer new formulations for separation axioms in *N*- topological space.

We present and study comparisons between *N*- topological space and Ann'a in the case of finite spaces.

2. Basic Definition of *N*- topological space and their properties

In this section we introduce the notion of *N*- topological space, and give its properties. Several of the classical results [2] are extended by defining appropriate substructures on the *N*- topological space. Examples are given to illustrate these structures.

Definition 2.1

Let (X, $\tau_1, \tau_2, \ldots, \tau_N$) be a non empty space with N different topology. { X, $\tau_1, \tau_2, \ldots, \tau_N$ } is called " ***N*- topological space** " if there exists N proper subspace $X_1, \ldots, X_N$ of X such that :

1. $X = X_1 \cup X_2 \cup \ldots \cup X_N$
2. $\tau_{X_i} = X_i \cap \tau_i$ is a subspace of (X, τ_i), where i =1,2,…,N .

Example 2.2

Let ({1, 2, 3}, τ_1, τ_2, τ_3) be 3 – topological space where τ_1={X, ∅} when X= {1,2,3}, τ_2={X, {1}, ∅} and τ_3={X,{2},∅}

Let X_1={1}, X_2={2} and X_3={3}.It clear that X=$X_1 \cup X_2 \cup X_3$ and τ_{X_1} = {{1}, ∅}, τ_{X_2}={{ 2},∅}, τ_{X_3}= {{3},∅}.

It is clear that $(X_i, \tau_{X_i})$is a topological space of (X, τ_i) for i=1,2,3.

Now, we give the definition of open set in *N*- topological space.

Definition 2.3

A subset U of *N*- topological space (X, $\tau_1, \tau_2, \ldots, \tau_N$) is said to be an "***N*- open set** " if and only if it is open in τ_i, for some i = 1,2,…,N.

Definition 2.4

The complement of *N*- open set in *N*- topological space (X , $\tau_1, \tau_2, \ldots, \tau_N$) is said to be an "***N*-closed set** ".

Remark 2.5

1- Every open set in topological space (X, τ_i) is *N*- open set, for all i= 1,2,…,N, But the converse is not true, (see the following Example 2.6).

2- Every closed set in topological space (X, τ_i) is *N*- closed set, for all i=1,2,..,N But the converse is not true, (see the following Example 2.6).

3- Every open set in subspace (X_i, τ_{X_i}), for all i= 1,2,…,N need not to be *N*- open set in *N*- topological space (X, $\tau_1, \tau_2, \ldots, \tau_N$), only if the subspace X_i is open in (X, τ_i), (see the following Example 2.6).

Example 2.6

Let $(\mathbb{N}, \tau_1, \tau_2, \tau_3, \tau_4)$ be 4-topological space (where $\mathbb{N}$ is the set of natural numbers) such that $\tau_1 = \{\mathbb{N}, \{1\}, \emptyset\}, \tau_2 = \{\mathbb{N}, \{2\}, \emptyset\}, \tau_3 = \{\mathbb{N}, \{3,4\}, \emptyset\}, \tau_4 = \{\mathbb{N}, \{5\}, \emptyset\}$.

And let X_1, X_2, X_3, X_4 be four subspace of X such that:

X_1= {1} implies $\tau_{X_1} = \{\{1\}, \emptyset\}$

X_2= {2} implies $\tau_{X_2} = \{\{2\}, \emptyset\}$

X_3= {3} implies $\tau_{X_3} = \{\{3\}, \emptyset\}$

X_4= {4, 5, 6, 7, …} implies $\tau_{X_4} = \{\{4, 5, 6, 7, \ldots\}, \{5\}, \emptyset\}$.

It is clear that $X = X_1 \cup X_2 \cup X_3 \cup X_4$

Now, to show the converse of part (1) is not true, let {1} is *N*- open set in $(\mathbb{N}, \tau_1, \tau_2, \tau_3, \tau_4)$ but it is not open set in each (X, τ_i); i= 1,2,…,N.

Also, to show the converse of part (2) is not true, let {2, 3, 4, 5, …} is *N*- closed set $(\mathbb{N}, \tau_1, \tau_2, \tau_3, \tau_4)$ but it is not closed set in each (X, τ_i); i= 1,2,…,N. And also notice that the subspace X_1= {1} is open in $(\mathbb{N}, \tau_1)$ implies that each open set in (X_1, τ_{X_1}) is *N*- open set in $(\mathbb{N}, \tau_1, \tau_2, \tau_3, \tau_4)$. But in the other hand notice that the subspace X_3= {3} is not open in (X_3, τ_3) implies that each open set in (X_3, τ_3) need not to be *N*- open set in $(\mathbb{N}, \tau_1, \tau_2, \tau_3, \tau_4)$.

Next, we give a definition about sub *N*- topological space.

Definition 2.7

Let (X, $\tau_1, \tau_2, \ldots, \tau_N$) be *N*- topological space ($N \geq 2$) . The subspace Y(≠ ∅) of X is called a sub *N*- topological space of X if and only if

(i) There exists N proper subspace $Y_1, Y_2, \ldots, Y_N$ of Y such that $Y_i \subseteq X_i$, $Y = Y_1 \sqcup Y_2 \sqcup \ldots \sqcup Y_N$ and (Y, $\tau_1', \tau_2', \ldots, \tau_N'$) is sub-*N*-topological space of (X, $\tau_1, \tau_2, \ldots, \tau_N$), that is $\tau_i' = Y \cap \tau_i$.

(ii) $\tau_{Y_i} = Y_i \cap \tau_i'$ is subspace of (Y , τ_i') , $i = 1,2,\ldots,N$.

Eexample 2.8

Let (X, τ_1, τ_2, τ_3) is 3 – topological space where X={1,2,3,4,5,6,7}, τ_1={X, ∅} , τ_2={X, ∅, {2,4,6}}, and τ_3={ X, ∅, {3,7}} and let $X_1 = \{1,2,3\}$, $X_2 = \{4,5\}$, $X_3 = \{6,7\}$. Let Y ={2,3,5,6}, $\tau_1' = Y \cap \tau_1$ ={Y, ∅}, $\tau_2' = Y \cap \tau_2$ = {Y, ∅, {2,6}}, $\tau_3' = Y \cap \tau_3$ = {Y, ∅, {3}}.

It is clear that (Y, $\tau_1', \tau_2', \tau_3'$) is sub 3- topological space of (X, τ_1, τ_2, τ_3) where $Y = Y_1 \sqcup Y_2 \sqcup Y_3$, Y_1={2,3} $\subseteq X_1$, Y_2={5} $\subset X_2$, Y_3={6} $\subseteq X_3$ and $\tau_{Y_1} = Y_1 \cap \tau_1' = \{\varnothing, Y_1\}$, $\tau_{Y_2} = Y_2 \cap \tau_2'$ $\{\varnothing, Y_2\}$, $\tau_{Y_3} = Y_3 \cap \tau_3'$ ={∅, Y_3 }.

Now, we introduce definitions and examples about separation axioms in *N*- topological space.

Definition 2.9

An *N*- topological space (X, $\tau_1, \tau_2, \ldots, \tau_N$) is said to be an "**N-T_o -space** " if and only if for each pair of distinct points x, y∈X, there exists *N*- open set U of X such that x∈U and y∉U.

Proposition 2.10

An *N*- topological space (X, $\tau_1, \tau_2, \ldots, \tau_N$) is *N*-T_o –space if (X , τ_i) is T_o –space for some i = 1,2,…,N.

Proof

To prove (X, $\tau_1, \tau_2, \ldots, \tau_N$) is *N* - T_o- space, we must prove for any x, y∈X such that x≠ y, there exists N-open set U of X such that x∈U and y∉U.

Now, let x, y∈X; x≠ y since there exists i∈1,2,…,N such that (X , τ_i) is T_o –space, implies there exists open set U in τ_i such that x∈U and y∉U, therefore ∃ N-open set U of X such that x∈U and y∉U(by Definition 2.3). Thus (X, $\tau_1, \tau_2, \ldots, \tau_N$) is *N* - T_o- space. ■

Remark 2.11

The converse of (Proposition 2.10) is not true, to see this, let ({1, 2, 3}, τ_1, τ_2, τ_3)be 3-topological space in (Example 2.2) , then the space ({1,2,3}, τ_1, τ_2, τ_3) is 3-T_o – space, but each (X , τ_1), (X , τ_2) , (X , τ_3) is not T_o- space.

Remark 2.12

If (X, $\tau_1, \tau_2, \ldots, \tau_N$) is *N* - T_o- space then (X_i, τ_{X_i}) need not to be T_o- space for all i= 1,2,…,N, only if (X , τ_i) is T_o –space.

Theorem 2.13

Let (X, $\tau_1, \tau_2, \ldots, \tau_N$) is *N* - T_o- space and (Y, $\tau_1', \tau_2', \ldots, \tau_N'$) is a sub *N*- topological space of the *N*- topological space (X, $\tau_1, \tau_2, \ldots, \tau_N$). Then(Y, $\tau_1', \tau_2', \ldots, \tau_N'$) is also *N*- T_o- space.

Proof

To prove (Y, $\tau_1', \tau_2', \ldots, \tau_N'$) is *N*- T_o- space, we must prove :∀x, y ∈Y, x≠ y, ∃U∈τ_i' for some i , such that x∈U and y∉U.

Now, let x, y $\in$Y, x$\neq$ y implies x, y $\in$X. Then, there exists W$\in\tau_i$ for some i , such that (x $\in$W $\wedge$ y $\notin$W) or (x $\notin$W $\wedge$ y $\in$W) since (X, $\tau_1, \tau_2, \dots, \tau_N$) is $N-T_S$- space.

Then Y$\cap$ W$\in\tau_i'$ for some i (by definition of τ_i')

So, x$\in$Y$\wedge$W $\Rightarrow$ x $\in$ Y$\cap$ W and y$\notin$W$\Rightarrow$ y$\notin$ Y$\cap$ W

Or x $\notin$W $\Rightarrow$ x $\notin$ Y$\cap$ W and y $\in$Y$\wedge$W $\Rightarrow$ y $\in$ Y$\cap$ W

Then (Y, τ_i') is T_S- space. Then (Y, $\tau_1', \tau_2', \dots, \tau_N'$) is N- T_S- space. ■

Example 2.14

Let ({1, 2, 3, 4}, τ_1, τ_2, τ_3) is 3-T_S- space where τ_1={X, $\emptyset$, {2}}, τ_2={X, $\emptyset$, {2}, {4},{2,4}} and τ_3={ X, $\emptyset$, {2,4},{3,4},{4},{2,3,4}}.

And let Y ={1,3,4}$\subset$X , Then (Y, $\tau_1', \tau_2', \tau_3'$) is sub 3-$T_S$- space where τ_1'={Y, $\varnothing$}, τ_2'={Y, $\varnothing$,{4}}, τ_3'={Y, $\varnothing$,{4},{3,4}}.

Definition 2.15

An N- topological space (X, $\tau_1, \tau_2, \dots, \tau_N$) is said to be an "***N*-*T*$_1$ -space** " if and if for each pair of distinct points x, y$\in$X, there exists two N- open sets U and V of X such that x$\in$U $\wedge$ y$\notin$U and x$\notin$V $\wedge$ y$\in$V.

Proposition 2.16

An N- topological space (X, $\tau_1, \tau_2, \dots, \tau_N$) is N-T_1 -space if (X , τ_i) is T_1 –space for some i = 1,2,…,N.

Proof

To prove (X, $\tau_1, \tau_2, \dots, \tau_N$) is N - T_1- space, we must prove for any x, y$\in$X such that x$\neq$ y, there exists two N-open sets U and V of X such that x$\in$U $\wedge$ y$\notin$U and x$\notin$V $\wedge$ y$\in$V.

Now, let x, y$\in$X; x$\neq$ y since there exists i$\in$1,2,…,N such that (X , τ_i) is T_1 – space, implies there exists two open sets U and V in τ_i such that x$\in$U $\wedge$ y$\notin$U and x$\notin$V $\wedge$ y$\in$V, therefore there exists two open sets U and V of X such that x$\in$U $\wedge$ y$\notin$U and x$\notin$V $\wedge$ y$\in$V(by Definition 2.3). Thus (X, $\tau_1, \tau_2, \dots, \tau_N$) is N-T_1 –space. ■

Remark 2.17

The converse of (Proposition 2.16) is not true, as the following example:

Example 2.18

Let X= {1,2,3,…,n}; n$\in\mathbb{N}$ and let τ_i={X, $\emptyset$, {i}}; i=1,2,3,…,n, then the space (X, $\tau_1, \tau_2, \dots, \tau_n$) is n-topological space, notice that (X, $\tau_1, \tau_2, \dots, \tau_n$) is N-T_1 -space but each (X, τ_i) is not T_1 –space, for all i.

Note

1. It is clear that each N- τ_1- space is also, N- T_S- space but the converse is not true see (Example 2.14), the space is 3-T_S- space but not 3- T_1- space.
2. If N- topological space is not $N-T_S$- space, then it is not N - T_1- space.

Theorem 2.19

Let (X, $\tau_1, \tau_2, \dots, \tau_N$) is N - T_1- space and (Y, $\tau_1', \tau_2', \dots, \tau_N'$) is sub N- topological space of (X, $\tau_1, \tau_2, \dots, \tau_N$). Then(Y, $\tau_1', \tau_2', \dots, \tau_N'$) is N- T_1- space.

Proof

To prove (Y, $\tau_1', \tau_2', \dots, \tau_N'$) is N- T_1- space, we must prove :$\forall$x, y $\in$Y, x$\neq$ y, $\exists$U,V$\in\tau_i'$ for some i , such that x$\in$U$\wedge$ y$\notin$U and x $\notin$ V$\wedge$ y$\in$V.

Now, since Y⊆ X, then x, y ∈X and X is N - T_1- space, then ∃$W_1, W_2 \in \tau_i$ for some i , such that x∈W_1∧ y ∉W_1 and x∉W_2∧ y ∈W_2.

Then Y∩W_1, Y∩ W_2 ∈ τ_i' for some i , (by definition of τ_i').

So x∈ Y∩W_1 ∧ y∉ Y∩W_1 and x∉ Y∩ W_2 ∧ y∈ Y∩ W_2.

Then (Y, τ_i') is T_1- space for some i , then (Y, $\tau_1', \tau_2', \dots, \tau_N'$) is N- T_1- space. ■

Definition 2.20

An N- topological space (X, $\tau_1, \tau_2, \dots, \tau_N$) is said to be an "***N*-T_2 -space** " if and if ∀x, y ∈X∧ x≠ y,∃U,V N-open sets of X, such that U∩V=∅ , x∈U ∧ y∈ V.

Proposition 2.21

An N- topological space (X, $\tau_1, \tau_2, \dots, \tau_N$) is N-T_2 -space if (X ,τ_i) is T_2 –space for some i = 1, 2, …, N.

Proof

To prove (X, $\tau_1, \tau_2, \dots, \tau_N$) is N - T_2- space, we must prove for any x, y∈X such that x≠ y, there exists two N-open sets U and V of X such that U∩V=∅ , x∈U ∧ y∈ V.

Now, let x, y∈X; x≠ y since there exists i∈1,2,…,N such that (X ,τ_i) is T_2 – space, implies there exists two open sets U and V in τ_i such that U∩V=∅ , x∈U ∧ y∈ V, therefore there exists two open sets U and V of X such that U∩V=∅ , x∈U ∧ y∈ V(by Definition 2.3). Thus (X, $\tau_1, \tau_2, \dots, \tau_N$) is N-T_2 –space. ■

Remark 2.22

The converse of (Proposition 2.21) is not true, to see that consider the n-topological space (X, $\tau_1, \tau_2, \dots, \tau_n$) in(Example 2.18), which is n-T_2 -space but each (X, τ_i) is not T_2 –space, for all i.

Note

1. It is clear that each N - T_2- space is also, N- T_1- space , so is N - T_o- space but the converse is not true.
2. If N- topological space is not N - T_o- space, then it is not N - T_1- space, then it is not N - T_2- space.

Theorem 2.23

Let (X, $\tau_1, \tau_2, \dots, \tau_N$) is N −T_2- space and (Y, $\tau_1', \tau_2', \dots, \tau_N'$) is sub N- topological space of (X, $\tau_1, \tau_2, \dots, \tau_N$). Then (Y, $\tau_1', \tau_2', \dots, \tau_N'$) is N- T_2- space.

Proof

The proof of this theorem is similar of the proof of theorem 2.19 ■

3. Application of N- topological space in ANN 'S

3.1. What are Artificial Neural Networks?

An Artificial Neural Network (ANN) is an information processing paradigm that is inspired by the way biological nervous systems, such as the brain, process information. The key element of this paradigm is the novel structure of the information processing system. It is composed of a large number of highly interconnected processing elements (neurons) working in unison to solve specific problems. ANN's, like people, learn by example. An ANN is configured for a specific application, such as pattern recognition or data classification, through a learning process. Learning in biological systems involves adjustments to the synaptic connections that exist between the neurons. This is true of ANN's as well.[3] That is Artificial Neural Networks are relatively crude electronic models based on the neural structure of the brain. The brain basically learns from experience. It is natural proof that some problems that are beyond the scope of current computers are indeed solvable by small energy efficient packages. This brain modeling also promises a less technical way to develop machine solutions. This new approach to computing also provides a more graceful degradation during system overload than its more traditional counterparts.

These biologically inspired methods of computing are thought to be the next major advancement in the computing industry. Even simple animal brains are capable of functions that are currently impossible for computers. Computers do rote things well, like keeping ledgers or performing complex math. But computers have trouble recognizing even simple patterns much less generalizing those patterns of the past into actions of the future.

3.2. Artificial Network Operations

The other part of the "art" of using neural networks revolves around the myriad of ways these individual neurons can be clustered together. This clustering occurs in the human mind in such a way that information can be processed in a dynamic, interactive, and self-organizing way. Biologically, neural networks are constructed in a three-dimensional world from microscopic components. These neurons seem capable of nearly unrestricted interconnections. That is not true of any proposed, or existing, man-made network. Integrated circuits, using current technology, are two-dimensional devices with a limited number of layers for interconnection. This physical reality restrains the types, and scope, of artificial neural networks that can be implemented in silicon.[3]

Currently, neural networks are the simple clustering of the primitive artificial neurons. This clustering occurs by creating layers which are then connected to one another.

How these layers connect is the other part of the "art" of engineering networks to resolve real world problems.

Basically, all artificial neural networks have a similar structure or topology as shown in Figure (1). In that structure some of the neurons interfaces to the real world to receive its inputs. Other neurons provide the real world with the networks outputs.

This output might be the particular character that the network thinks that it has scanned or the particular image it thinks is being viewed. All the rest of the neurons are hidden from view.

But a neural network is more than a bunch of neurons. Some early researchers tried to simply connect neurons in a random manner, without much success. Now, it is known that even the brains of snails are structured devices. One of the easiest ways to design a structure is to create layers of elements. It is the grouping of these neurons into layers, the connections between these layers, and the summation and transfer functions that comprise a functioning neural network. The general terms used to describe these characteristics are common to all networks.

Although there are useful networks which contain only one layer, or even one element, most applications require networks that contain at least the three normal types of layers - input, hidden, and output. The layer of input neurons receives the data either from input files or directly from electronic sensors in real-time applications.

The output layer sends information directly to the outside world, to a secondary computer process, or to other devices such as a mechanical control system. Between these two layers, there can be many hidden layers. These internal layers contain many of the neurons in various interconnected structures.
The inputs and outputs of each of these hidden neurons simply go to other neurons.

In most networks each neuron in a hidden layer receives the signals from all of the neurons in a layer above it, typically an input layer. After a neuron performs its function it passes its output to all of the neurons in the layer below it, providing a feed forward path to the output. These lines of communication from one neuron to another are important aspects of neural networks. They are the glue to the system. They are the connections which provide a variable strength to an input. There are two types of these connections. One causes the summing mechanism of the next neuron to add while the other causes it to subtract. In more human terms one excites while the other inhibits.

Some networks want a neuron to inhibit the other neurons in the same layer. This is called lateral inhibition. The most common use of this is in the output layer. For example in text recognition if the probability of a character being a "P" is .85 and the probability of the character being an "F" is .65, the network wants to choose the highest probability and inhibit all the others. It can do that with lateral inhibition. This concept is also called competition.

3.3. A Relation Between ANN's and N- Topological space

ANN's have been developed as generalizations of mathematical models of human cognition or neural biology, and is characterized by :

1. It is a pattern of connections between the neurons and the layers (called topology of network).
2. It is a method of determining the weights on the connections (called it's training, or learning algorithm).
3. It is activation function.

A neural network consists of a number of simple processing elements called neurons, these neurons consist in many layer. The numbers of neurons and layers in the ANN's differ from not work to network and this is called the topology of the network .
The ANN's with multilayer is not well understood [4]. Some authors [5] see that little theoretical gain in using more than one hidden layer since a single hidden layer model suffices for density. In this paper, we introduce the definition and properties of *N*- topological space which can be applied to ANN's with more than one hidden layer.

One important advantage of the multiple layer model (N- topological space (see figure (2)) has to do with the existence of locally supported functions in the two hidden layer model (4- topological space) since for any activation function σ and every $g \neq 0$

$$g(x) = \sum_{i=1}^{r} c_i \, \sigma(w_i x \quad \theta_i), \; c_i, \theta_i \in R \text{ and } w_i \in R^n, \text{has} \int_{R^n} \|g(x)\|^p \, dx = \infty.$$

For every $p \in [1, \infty)$, and thus g(x) defined above has no compact support.(see [6], [7] for amore detailed discussion).
Another advantage of the multilayer model (*N*- topological space), there is a lower bound on the degree to which the single hidden layer model (3- topological space) with r neuron units in the hidden layer can approximate any function. It is given by the extent to which a linear combination of r activation function can approximate this same function, and, more importantly, activation function approximate itself is bounded below (a way from zero)with some non- trifling dependence on r and on the set to be approximated. In the single hidden layer model (3- topological space) there is an intrinsic lower bound on the degree of

approximation depending on the number of neuron units used. This is not the casa in the two hidden layer model, (4- topological space) .

Finally, we can show, using the kolmogorov super position theorem [7] that a finite number of units in both hidden layers (4- topological space) is sufficient to approximate arbitrarily well any continuous function.

Theorem 3.4

There exists an activation function σ which is C^{∞}, strictly increasing, and sigmoidal, and has the following property. For any $f \in C[0,1]^n$ and $\varepsilon > 0$, there exist constants d_i, c_{ij}, θ_{ij}, γ_i and vectors $W_{ij} \in R^n$, for which:

$$\left| f(x) - \sum_{i=1}^{2n+1} d_i \sigma\left(\sum_{j=1}^{2n+1} c_{ij} \sigma(W_{ij} x + \theta_{ij}) + \gamma_i \right) \right| < \varepsilon, \text{ for all } x \in [0,\ 1]^n.$$

Proof

Let f be any continuous function on $[0,1]^n$, $\varepsilon > 0$,then by Kolmogorov Superposition theorem , there exist constants c_i, θ_{ij} and vectors $W_{ij} \in R^n$, such that

$$\left| f(x) - \sum_{i=1}^{2n+1} d_i \Phi(v_{ij} x + s_{ij}) \right| < \varepsilon/(2n+1) \qquad \text{........(1)}$$

since ϕ is continuous function ,and by restrict ϕ in $[0,1]^n$ can represent ϕ such that

$$\phi(x) = \sum_{j=1}^{2n+1} c_{ij} \sigma(r_{ij} x + s_{ij}) \qquad \text{...........(2)}$$

By substituting (2) in (1) , we obtain :

$$\left| f(x) - \sum_{i=1}^{2n+1} d_i \sigma\left(\sum_{j=1}^{2n+1} c_{ij} \sigma(r_{ij}(v_{ij} x + s_{ij}) + L_{ij}) \right) \right| < \varepsilon$$

Then, $$\left| f(x) - \sum_{i=1}^{2n+1} d_i \sigma\left(\sum_{j=1}^{2n+1} c_{ij} \sigma(W_{ij} x + \theta_{ij}) + \gamma_i \right) \right| < \varepsilon \quad \square$$

Now, we introduce the following definition :

Definition 3.5

A set of functions is said to be fundamental in a given space if a linear combinations of them are dense in that space.

Theorem 3.6

Let K be a compact set in R^n. Then the set E of functions of the form $\mu(x) = \exp(a^T x)$, where $a \in R^n$, is fundamental in C(K).

Proof

By the Stone-Weierstrass theorem we need only show that the set forms an algebra and separates points. Suppose $x \in K$. First, we have:

$\exp(a^T x)\ \exp(b^T x) = \exp(a^T x + b^T x) = \exp((a^T + b^T)\ x\ )$.

The set also contains the function “1” simply choose $a = 0$. This establishes that E is an algebra. It remains to show that E separates the points of K. So let $x, y \in K$ with $x \neq y$. Set $a = (x - y)$. Then $a^T(x - y) \neq 0$, so $a^T x \neq a^T y$. Thus $\exp(a^T x) \neq \exp(a^T y)$.

The proof is complete. □

Before considering more constructive versions of this result we complete the density proof.

Theorem 3.7

Let K be a compact set in R^n. Then the set F of functions of the form g(x), defined by: $g(x) = \sum_{j=1}^{k} v_j \sigma(W_j^T x + c_j)$,with σ as a continuous sigmoidal function is dense in C(K).

Proof

Let $f \in C(K)$. For any $\varepsilon > 0$, there exists (by theorem 3.6) a finite number m of vectors a_i, such that:

$$\left\| f - \sum_{i=1}^{m} \exp\left(a_i^T x\right) \right\|_\infty < \frac{\varepsilon}{2}$$

since there are only m scalars $a_i^T x$, we may find a finite interval including all of them. Thus there exists a number Γ such that $\exp(a_i^T x) = \exp(\Gamma y)$, where $y = (a_i^T x/\Gamma) \in [0,1]$. Then theorem 3.6 tells us that the function exp (Γy) can be approximated by linear combinations functions of the form $\sigma(W_j^T x + c_j)$ with a uniform error less than $\varepsilon/2m$, from which the desired result easily follows. ■

Remarks

1. Theorem 3.6 tells us one hidden layer is sufficient to approximate any continuous function to any required accuracy.
2. Γ in the proof of theorem 3.7 can be chosen to be an integer .
3. The question of rate of convergence of approximations is obviously of considerable importance. If f is smooth and we use smooth approximating functions we might hope to get better convergence than the simple O(1/n) .

4. Conclusions

1. We define *N*- topological space and give some examples and properties about this notion.
2. We give application of *N*- topological space in ANN's, then we obtain :
   (i) Increasing number of hidden units leads to decreasing number of epoch of training.
   (ii) Large number of hidden units leads to a small error on the training set but not necessarily leads to a small error on the test set.
   (iii) If we fix the number of basis functions and increase the number of the layers of the ANN's (that's increase N in the *N*- topological space), then we get an accurate numerical solution.

References

1. EL – Fattah El- Atik, A. ; Abd El-Monasef, M. E. and Lashin, E.I. (2002) "On finite to topological aspaces, Ninth prague topological symposium, topology atlas, Toronto, PP.75-90.
2. Wilkins, D.R. (1998-1999)," Topological space ", Academic Year.

3. Stergiou, C. and Siganos, D. (2008) NEURAL NETWORKS , IEEE Neural Networks Council

4. Tawfiq, L.N.M. (2007), Density and Approximation by using Feed Forward Artificial Neural Networks, Ibn Al – Haithem journal for applied and pure science,.20, 1.

5. Dijkstra, E.W. (2001) "Approximation with Artificial Networks", Msc thesis, Faculty of mathematics and computing science Eindhoven university of technology, The Netherlands.
6. .Mason, J.C. and Parks, P.C. (2004) "Selection of neural network structures some approximation theory guidelines, IEE Control engineering series, 46:151-180.
7. Ellacott, S.W. (1994), "Aspects of the numerical analysis of neural networks", Acta numerical, 145-202.
8. Kurkova ,V. (1991)"Kolmogorov's theorem is relevant , 1," Neural computation 3: 617-622.

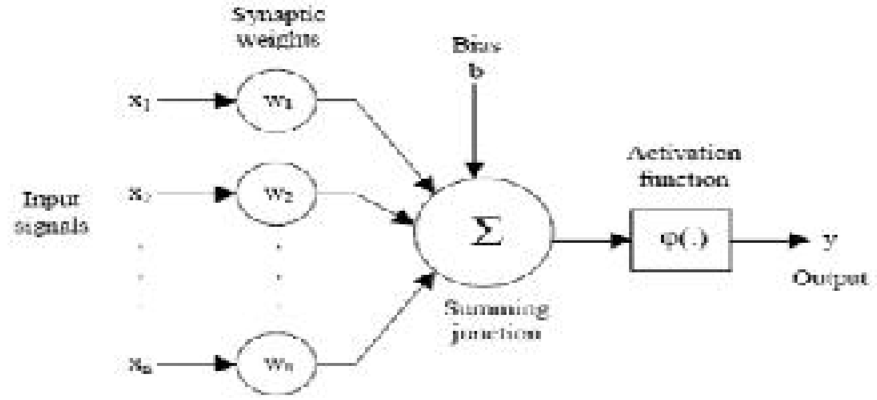

Fig. (1) A Simple Neural Network Diagram.

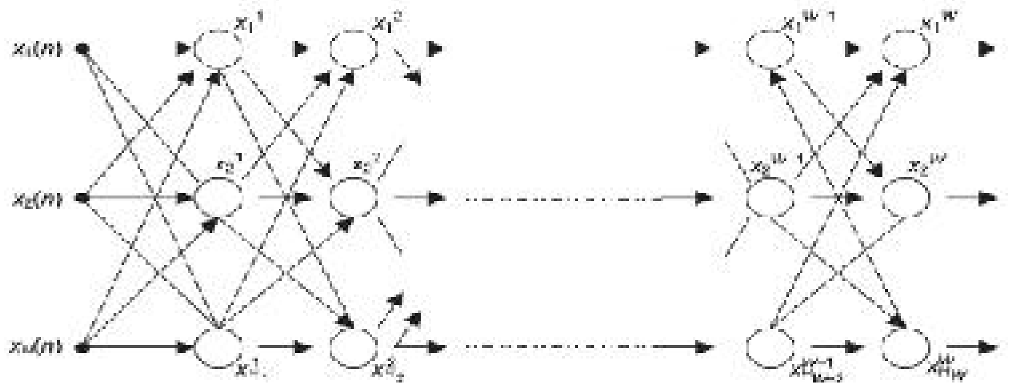

Fig. (2) Graph of a multilayer(N-Topological space)neural network with sequential connections

الفضاءات التبولوجية ـ N وتطبيقاتها في الشبكات العصبية الصناعية

لمى ناجي محمد توفيق و رشا ناصر مجيد

قسم الرياضيات ،كلية التربية – أبن الهيثم ،جامعة بغداد

استلم البحث في 16 آذار 2009

قبل البحث في 29 آذار 2010

الخلاصة

في هذا البحث أعطينا تعار يف خواص وأمثلة حول مصطلح الفضاءات التبولوجية من النوع .N . إذا إن N، في هذا البحث هو عدد منتهي موجب ، $N \geq 2$. إن فكرة هذا البحث هو دراسة وتقديم بعض خواص هذه الفضاءات مع دراسة العلاقة بين هذه الفضاءات والشبكات العصبية الصناعية وهذا يعني تطبيق تعريف هذا الفضاء في حقل الكومبيوتر ولاسيما في مجال المعالجات المتوازية.