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الخلاصة

الغرض من هذا البحث دراسة أنواع جديدة من التراص في الفضاءات التبلوجيه الثنائية، أذ سنقدم التراص من النوع– ال

L- compact Spaces

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Abstract

The purpose of this paper is to study a new types of compactness in bitopological spaces. We shall introduce the concepts of L- compactness.

Introduction

The concept of bitopological space was initiated by Kelly[1].A set X equipped with two Topologies τ_1 and τ_2 is called a bitopological space denoted by (X,τ_1,τ_2).

By a directed set we mean a pair (A, ≥) consisting of a non-empty set A and a binary relation ≥ defined on A and satisfies the following conditions:

(1) $a \geq a$ for each $a \in A$.

(2) If $a \geq b$ and $b \geq c$, then $a \geq c$ for each a, b, and c in A.

(3) For each two members a and b of A, there exists a member $c \in A$ such that $c \geq a$ and $c \geq b$.

If (A, ≥) is a directed set and f is a function of A into a non-empty set X, then f is called a "net" in X and is denoted by (f, X, A, ≥). The image of $a \in A$ under f is denoted by f_a and a net in X will be sometimes denoted by $\{f_a: a \in A\}$.[2]

A "filter" on a non-empty set X is a non-empty family F of subsets of X with the following properties:

(1) $\varnothing \notin F$.

(2) If $F \in F$ and $F \subseteq H$, then $H \in F$.

(3) If $F \in F$ and $H \in F$, then $F \cap H \in F$.

A filter on a non-empty set is said to be an ultrafilter if and only if it is not properly contained in any other filter on this set.[2]

L-open set was studied by Al-swid[2], asubset G of a bitopological space (X,τ_1,τ_2) is said to be "L –open" set if and only if there exists a τ_1-open set U such that $U \subseteq G \subseteq cl\tau_2(U)$,the family of all L-open subsets of X is denoted by L-O(X).The complement of an L-open set is called "L-closed" set,the family of all L-closed subsets of X is denoted by L-C(X).In a bitopological space (X,τ_1,τ_2) every τ_1-open set is an L-open set[3].The union of any family of L-open subsets of X is an L-open set, but the intersection of any two L-open subsets of X need not be L-open set[2].Al-Talkahny [3],introduced two new concepts "L-T_2 -spaces" and "L-continuous functions ". A bitopological space (X,τ_1,τ_2) is said to be "L-T_2 -space" if and only if for each pair of distinct points x and y in X,there exist two disjoint L-open subset G and H of X such that $x \in G$ and $y \in H$ .Let (X,τ_1,τ_2),(Y,τ_1',τ_2') be any bitopological spaces and let $f : X \to Y$ be any function, then f is said to be "L-continuous" function if and only if the inverse image of any L-open subset of Y is an L-open subset of X.

2- L-compactness

Definition(2.1)

Let (X,τ_1,τ_2) be a bitopological space and let A be a subset of X. By an "L-open cover of A" we mean a subcollection of the family L-O(X) which covers A.

Remark(2.2):

Every τ_1-open cover in a bitopological space (X,τ_1,τ_2) is an L-open cover.

The converse of remark (2.2) is not true in general as the following example shows:

Example (2.3)

$X = \{1,2,3,4\}$
$\tau_1 = \{X,\phi,\{1\},\{2,3\},\{1,2,3\}\}$
$\tau_2 = \{X,\phi,\{1\}\}$
$F_2 = \{X,\phi,\{2,3,4\}\}$

$L-O(X) = \{X,\phi,\{1\},\{2,3\},\{1,2,3\},\{1,2,4\},\{1,4\},\{1,2\},\{1,3,4\},\{1,3\},\{2,3,4\}\}$ Let $C = \{\{1\},\{2,3,4\}\}$, note that C is an L-open cover of X, but it is not τ_1-open cover.

Definition(2.4)

A bitopological space (X,τ_1,τ_2) is said to be "L-compact space" if and only if every L-open cover of X has a finit subcover.

Proposition (2.5)

If a bitopological space (X,τ_1,τ_2) is an L-compact space, then (X,τ_1) is a compact space.

Proof: follows from remark (2.2).

Remark (2.6)

The opposite direction of proposition (2.5) is not true in general, as the following example shows:

Let $X = \mathrm{N}$ and let $x_o \in \mathrm{N}$

$\tau_1 = \{\mathrm{N}, \phi, \{x_o\}\}$

$\tau_2 = I$ =The indiscrete topology

$L-O(X)=\{U \subseteq \mathrm{N}; x_o \in U \text{ or } U=\phi\}$

Note that (N, τ_1) is compact but $(\mathrm{N}, \tau_1, \tau_2)$ is not L-compact.

Proposition (2.7)

An L-closed subset of an L-compact space is L-compact.

Proof:

Let A be an L-closed subset of an L-compact space (X,τ_1,τ_2) and let $\{G_\alpha : \alpha \in \Lambda\}$ be an L-open cover of A .Then $\{G_\alpha : \alpha \in \Lambda\} \cup A^c$ forms an L-open cover of X which is L-compact space. So there are finitely many elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $X = \bigcup_{i=1}^{n} G_{\alpha_i} \cup A^c$,it follows that $A \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$ .Hence A is an L-compact.

Corollary (2.8)

An L-closed subset of an L-compact space (X,τ_1,τ_2) is τ_1-compact.

Proof:

Follows from proposition (2.7) and (2.5).

Corollary (2.9)

A τ_1-closed subset of an L-compact space (X,τ_1,τ_2) is L-compact.

Proof:

Since every τ_1-closed set is an L-closed set and by proposition (2.7).

Corollary (2.10)

A τ_1-closed subset of an L-compact space (X,τ_1,τ_2) is τ_1-compact.

Proof:

Follows from corollary(2.9) and proposition (2.5).

Proposition(2.11)

The L-continuous image of an L-compact space is an L-compact.

Proof:

Suppose that $f:(X,\tau_1,\tau_2)\rightarrow(Y,\tau_1,\tau_2)$ is an L-continuous and onto function and X is an L-compact space. Let $\{G_\alpha:\alpha\in\Delta\}$ be an L-open cover of Y , it follows that $\{f^{-1}(G_\alpha):\alpha\in\Delta\}$ is an L-open cover of X which is L-compact.So there are finitely many elements $\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n$ such that $X=\bigcup_{i=1}^{n}f^{-1}(G_{\alpha_i})=f^{-1}\left(\bigcup_{i=1}^{n}G_{\alpha_i}\right)$ .Therefore $Y=\bigcup_{i=1}^{n}G_{\alpha_i}$,hence Y is an L-compact.

Corollary (2.12)

Let $f:(X,\tau_1,\tau_2)\rightarrow(Y,\tau_1,\tau_2)$ be an L-continuous function, then $f(A)$ is a compact subset of (Y,τ_1) for each L-compact subset A of X .

Proof:

Follows from propositions (2.11) and (2.5).

It is known that every compact subset of any T_2 -space is closed. If we change the concepts of compact, T_2 and closed by the concepts L-compact-T_2 and L-closed, then this fact being invalid in general, as the following example shows:

Example (2.13)

$X=\{1,2,3\}$
$\tau_1=\{X,\phi,\{1\},\{2\},\{1,2\}\}$
$\tau_2=I$

$$L-O(X)=\{X,\phi,\{1\},\{2\},\{1,2\},\{1,3\},\{2,3\}\}$$

$L-C(X)=\{X,\phi,\{2,3\},\{1,3\},\{3\},\{2\},\{1\}\}$. Clear that X is an L-T_2-space. If $A=\{1,2\}$,then A is an L-compact subset of X, but it is not L-closed.

Definition (2.14): [3]

Let (X,τ_1,τ_2) be a bitopological space and let A be a subset of X, $x\in X$ .Then A is called an

L-neighborhood of x if and only if there is an L-open set G in X such that $x \in G \subseteq A$.

Definition (2.15) [3]

Let (X, τ_1, τ_2) be a bitopological space and let A be a subset of X.The intersection of all L-closed set containing A is called "L-closure of A"denoted by L-cl(A).

Theorem (2.16) [4]

Let (X, τ_1, τ_2) be a bitopological space and let A be a subset of X. A point x in X is an L-closure point of A if and only if every L-open neighborhood of x intersects A.

Definition (2.17) [4]

Let (X, τ_1, τ_2) be a bitopological space and let $(f, X, A, \geq)$ be a net in X ,then f is said to be "L-convergent "to a point x_o in X if and only if for each L-open neighborhood N of x_o,there exists an element $a_o \in A$ such that $f_a \in N$ for each $a \geq a_o$.

Definition (2.18) [4]

Let (X, τ_1, τ_2) be a bitopological space and let $(f, X, A, \geq)$ be a net in X. A point x_o in X is called an "L-cluster point of f" if and only if for each $a \in A$ and for each L-open neighborhood N of x_o,there exists an element $b \geq a$ in A such that $f_b \in N$.

Theorem (2.19) [4]

Let (X, τ_1, τ_2) be a bitopological space and let $(f, X, A, \geq)$ be a net in X. For each $a \in A$ let $M_\alpha = \{f(x) : x \geq a \, in \, A\}$,then a point p of X is an L-cluster point of f if and only if $p \in L - cl(M_\alpha)$ for each $a \in A$.

Definition (2.20)

Let (X, τ_1, τ_2) be a bitopological space and let Ғ be a filter on X . A point x in X is called an "L-cluster point of Ғ" if and only if each L-open neighborhood of x intersects every member of Ғ.

Theorem (2.21)

Let (X, τ_1, τ_2) be a bitopological space and let Ғ be a filter on X . A point p in X is an L-cluster point of Ғ if and only if $p \in L - cl(F)$ for each $F \in$ Ғ.

Proof: the "first direction"

Suppose that p is an L-cluster point of Ғ. then for each L-open neighborhood G of p, $G \cap F \neq \phi$ for each $F \in$ Ғ, it follows by theorem (2.16) that $p \in L - cl(F)$ for each $F \in$ Ғ.

The "second direction"

Assume that $p \in L-cl(F)$ for each $F \in$ F, then by theorem (2.16) every L-open neighborhood of p intersects F for each $F \in$ F. Hence p is an L-cluster point of F

Definition (2.22) [2]

A collection of sets is said to have the finite intersection property (FIP) if and only if the intersection of each finite subcollection of it is non empty.

Remark (2.23) [2]

Every filter in a non- empty set X has the FIP.

Theorem (2.24) [3]

Let A be a non empty collection of subsets of a set X such that A has the FIP. Then there exists an ultra filter F containing A .

Proposition (2.25) [4]

Let A be a subset of a bitopological space (X,τ_1,τ_2). Then A is an L-closed set if and only if $A = L-cl(A)$.

Theorem (2.26)

Let (X,τ_1,τ_2) be a bitopological space. Then the following statements are equivalent:

1- X is an L-compact space,

2- Every collection of L-closed subsets of X with the FIP has anon empty intersection, and

3- Every filter on X has an L-cluster point.

Proof:

1→2

Let $\{F_\alpha : \alpha \in \Lambda\}$ be a collection of L-closed subset of X with the FIP. suppose that $\bigcap_{\alpha\in\Lambda} F_\alpha = \phi$,

it follows by De-Morgan Laws that $\bigcup_{\alpha\in\Lambda} F_\alpha^c = X$ therefore $\left\{F_\alpha^c : \alpha \in \Lambda\right\}$ forms an L-open cover for X which is an L-compact space, then there exists finitely many elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\bigcup_{i=1}^{n} F_{\alpha_i}^c = X$. Again by De-Morgan Laws we have that $\bigcap_{i=1}^{n} F_{\alpha_i} = \phi$ which is a contradiction since $\{F_\alpha : \alpha \in \Lambda\}$ has the FIP. Hence $\bigcap_{\alpha\in\Lambda} F_\alpha = \phi$

2→3

Let $\mathcal{F}$ be a filter on X, then by remark (2.23) $\mathcal{F}$ has the FIP, it follows that the collection $\{L-cl(F): F \in \mathcal{F}\}$ of L-closed subsets of X also has the FIP, so by (2) there exists at least one point $x \in \bigcap\{L-cl(F): F \in \mathcal{F}\}$ then by theorem (2.21) x is an L-cluster point of $\mathcal{F}$. Thus every filter on X has an L-cluster point.

3→1

Assume that every filter on X has an L-cluster point and let $\Im$ be an L-open cover of X. suppose ,if possible, $\Im$ has no finite sub cover the collection $\{X-G: G \in Y\}$ has the FIP, for if there is a finite sub collection $\{X-G_i : 1 \le i \le n\}$ of such that $\bigcap\{X-G_i : 1 \le i \le n\} = \phi$ this implies that $\bigcup\{G_i : 1 \le i \le n\} = X$ which contradicts our supposition that $\Im$ has no finite sub cover, thus must have the FIP, it follows by theorem (2.24)that there exists an ultra filter $\mathcal{F}$ on X containing .by (3) $\mathcal{F}$ has an L-cluster point $x \in X$, then by theorem (2.21) $x \in L-cl(F)$ for each $F \in \mathcal{F}$, in particular $x \in L-cl(X-G)$ for each $G \in \Im$. But X-G is an L-closed subset of X for each $G \in \Im$, therefore by propostion (2.25) $L-cl(X-G) = X-G$ for every $G \in \Im$. This implies $x \in \bigcap\{X-G : G \in \Im\}$, so by De-Morgen Laws $x \in X - \bigcup\{G : G \in \Im\}$,that is, $x \notin \bigcup\{G : G \in \Im\}$,which is a contradiction with the fact that $\Im$ is an L-open cover of X ,hence $\Im$ must have a finite sub cover and consequently X is an L-compact space.

Proposition (2.27):

Let (X, τ_1, τ_2) be a bitopological space. If X is an L-compact space, then every net in X has an L-cluster point.

Proof:

let $(f, X, A, \le)$ be a net in X . for each $a \in A$ let $K_a = \{f_x : x \ge a \text{ in } A\}$. Since A is directed by $\ge$,so the collection $\{K_a : a \in A\}$ has the FIP. Hence $\{L-cl(K_a): a \in A\}$ also has the FIP , it follows by theorem (2.26) $\bigcap_{a \in A} L-cl(K_a) \ne \phi$ let $p \in \bigcap_{a \in A} L-cl(K_a)$, then $p \in L-cl(K_a)$ for each $a \in A$,so by theorem (2.19) p is an L-cluster point of f.

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