ملاحظات حول معادلة المؤثراللاخطية $X + A^* X^{-n} A = I$

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الخلاصة

الشروط الضرورية والكافية لمعادلة المؤثر $X + A^* X^{-n} A = I$ للحصول على حل موجب حقيقي ذاتي الترافق X قد اعطيت بالاعتماد على هذه الشروط وبعض الخصائص للمؤثر، وكذلك العلاقة بين الحل X و A قد اعطيت أيضا.

الكلمات المفتاحية: معادلة المؤثر اللاخطية،القطر الطيفي، مؤثر موجب ذاتي الترافق

Notes On The Non Linear Operator Equation

$X + A^{*}X^{-n}A = I$

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Abstract

Necessary and sufficient conditions for the operator equation $X + A^{*}X^{-n}A = I$, to have a real positive definite solution X are given. Based on these conditions, some properties of the operator A as well as relation between the solutions X and A are given.
Key words: non-linear operator equation; spectral radius; positive definite operator.
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Introduction

Consider the non-linear operator equation

$$X + A^{*}X^{-n}A = I \qquad (1)$$

where I is identity operator, and $A, A^{*}, X \in B(H)$; where $B(H)$ denotes the Banach algebra of all bounded linear operators on H; H is an infinite dimensional complex Hilbert space. Several authors have studied the above equation when A, X are matrices and $n = 1$, $n = 2$ and they have obtained theoretical properties of these equations. In [1] Equation (1) was studied in the case X is a self_adjoint positive operator , which arises in many applications such as in control theory and statistics and in dynamic programming

In this paper, we study equation (1) where X belongs to the set; where

$$C := \{A \mid A = T^{*}T \mid, T \in B(H); r(T) = \|T\|\}.$$

Where $r(T)$ is the spectral radius of T

1-Preliminaries

In this section we present notation, lemma and theorem which will be used in the remainder of the paper. The notation $A > 0$ $(A \geq 0)$ means that A is positive operator , and $A > B$ is used as an alternative notation for $A - B > 0$ .It is well-known for any operator $T \in B(H), T^{*}T$ is positive operator $[2, p.22]$,let spec A denotes the spectrum of A.

Lemma 1.1[3, p. 866]: Let M and N be two arbitrary operators then:

$$r(M^{*}N - N^{*}M) \leq r(M^{*}M + N^{*}N)$$

Proof: By elementary calculus, we have that

$$r(M^{*}N - N^{*}M) = r\left(\begin{bmatrix} M^{*} & N^{*} \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & O \end{bmatrix} \begin{pmatrix} M \\ N \end{pmatrix} \right)$$

Since the non-zero elements of spec MN and spec NM are the same [4, P.43]; so for any two operators, we have:

$$r\left(\begin{bmatrix} M^* & N^* \end{bmatrix}\begin{bmatrix} O & I \\ -I & O \end{bmatrix}\begin{bmatrix} M \\ N \end{bmatrix}\right) = r\left(\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{pmatrix} M \\ N \end{pmatrix}\begin{pmatrix} M^* & N^* \end{pmatrix}\right)$$

Now, $r(A) = \|A\|$, where $\| \ \|$ denotes the operator norm. so

$$r\left(\begin{bmatrix} O & I \\ -I & O \end{bmatrix}\begin{pmatrix} M \\ N \end{pmatrix}\begin{pmatrix} M^* & N^* \end{pmatrix}\right) = r\left\|\begin{pmatrix} 0 & I \\ -I & O \end{pmatrix}\begin{pmatrix} M \\ N \end{pmatrix}\begin{pmatrix} M^* & N^* \end{pmatrix}\right\|$$

$$\leq \left\|\begin{bmatrix} O & I \\ -I & O \end{bmatrix}\right\| \left\|\begin{bmatrix} M \\ N \end{bmatrix}\begin{bmatrix} M^* & N^* \end{bmatrix}\right\|$$

$$\leq 1\ r\left(\begin{bmatrix} M \\ N \end{bmatrix}\begin{bmatrix} M^* & N^* \end{bmatrix}\right)$$

$$\leq r\left(\begin{bmatrix} M^* & N^* \end{bmatrix}\begin{bmatrix} M \\ N \end{bmatrix}\right)$$

$$\leq r\left(M^*M + N^*N\right)$$

Which completes the proof.

2- Necessary and sufficient conditions of the solution of the equation

We study the existence of the solution of equation (1) by the following theorem:

Theorem 2.1: the operator equation (1) has a solution X positive operator if and only if the operator A takes the following factorization form

$$A = \begin{cases} \left(W^*W\right)^{\frac{n-1}{2}} W^* Z & \ldots \textit{if } n \textit{ is odd} \\ \left(W^*W\right)^{\frac{n}{2}} Z & \ldots \textit{if } n \textit{ is even} \end{cases} \qquad (2)$$

where W is an invertible operator and $W^*W + Z^*Z = I$.

Proof: suppose that equation (1) has a solution X . Then, using the set C we can write X as $X = W^*W$.

Equation (1) can be written as

$$W^*W + A^*\left(W^*W\right)^{-n} A = I$$

The prove using mathematical induction:

- Suppose $n = 1$, then

$$W^*W + A^*\left(W^*W\right)^{-1} A = I$$

$$W^*W + A^*W^{-1}\left(W^*\right)^{-1} A = I$$

Further, we can rewrite the last equations as:

$$W^*W + \left(\left(W^{-1}\right)^* A\right)^* \left(W^*\right)^{-1} A = I \qquad (3)$$

Equation (3) can be rewritten in the equivalent form [5, p.171]:

$$\begin{bmatrix} W \\ W^{-*}A \end{bmatrix}^* \begin{bmatrix} W \\ W^{-*}A \end{bmatrix} = I \qquad (4)$$

Now, set $Z = W^{-*}A$; then $A = W^* Z$ as desired,

- Suppose it is true when $n = p$ to show that it is true when $n = p+1$

$$W^*W + A^*\left(W^*W\right)^{-(P+1)}A = I$$

$$W^*W + A^*\left(W^*W\right)^{-P}\left(W^*W\right)^{-1}A = I$$

If p is odd,

$$W^*W + A^*\left(W^*W\right)^{-1}\left(W^*W\right)^{-1}\left(W^*W\right)^{-1}\ldots\left(W^*W\right)^{-1}\left(W^*W\right)^{-1}A = I$$

then $W^*W + A^*W^{-1}W^{-*}W^{-1}\ldots W^{-1}W^{-*}W^{-1}W^{-*}A = I$

$$W^*W + \left(W^{-*}W^{-1}W^{-*}W^{-1}\ldots W^{-*}A\right)^*\left(W^{-*}W^{-1}W^{-*}\ldots W^{-*}\right)A = I \qquad (5)$$

Equation (5) can be rewritten in the equivalent form:

$$\begin{bmatrix} W \\ W^{-*}W^{-1}W^{-*}\ldots W^{-*}A \end{bmatrix}^* \begin{bmatrix} W \\ W^{-*}W^{-1}W^{-*}\ldots W^{-*}A \end{bmatrix}$$

Now, set $Z = W^{-*}W^{-1}W^{-*}\ldots W^{-*}A$, then $A = W^*W\ W^*W\ldots W^*Z$, as form $\left(W^*W\right)^{\frac{p-1}{2}}W^*\ Z$

If p is even, then:

$$W^*W + A^*\left(W^*W\right)^{-1}\left(W^*W\right)^{-1}\ldots\left(W^*W\right)^{-1}\left(W^*W\right)^{-1}A = I$$

$$W^*W + A^*W^{-1}W^{-*}W^{-1}W^{-*}\ldots W^{-1}W^{-*}W^{-1}W^{-*}A = I$$

$$W^*W + \left(W^{-1}W^{-*}W^{-1}\ldots W^{-*}A\right)^*\left(W^{-1}W^{-*}W^{-1}\ldots W^{-*}A\right) = I \qquad (6)$$

Equation (6) can be rewritten in the equivalent form:

$$\begin{bmatrix} W \\ W^{-1}W^{-*}W^{-1}\ldots W^{-*}A \end{bmatrix}^* \begin{bmatrix} W \\ W^{-1}W^{-*}W^{-1}\ldots W^{-*}A \end{bmatrix} = I$$

New, set $Z = W^{-1}W^{-*}W^{-1}\ldots W^{-*}A$; then $A = \left(W^*W\ \ W^*W\ \ W^*W\ldots W^*W\right)Z$, as form $\left(W^*W\right)^{\frac{P}{2}}Z$

Conversely,assume that the operator A admits the factorization $A = \left(W^*W\ W^*W\ldots W^*\right)Z$, if n is odd, and set $X = W^*W$, we then need to show that X (which is positive operator) is a solution to the operator equation (1), we have:

$$X + A^*X^{-n}A = W^*W + \left(W^*W\,W^*W\ldots W^*Z\right)^*\left(W^*W\right)^{-n}\left(W^*W\,W^*W\ldots W^*\right)Z$$

$$= W^*W + Z^*W\,W^*W\,W^*\ldots\left(W^*W\right)^{-1}\ldots\left(W^*W\right)^{-1}\left(W^*W\,W^*W\ldots W^*\right)Z$$

$$= W^*W + Z^*W\,W^*\ldots W\,W^{-1}W^{-*}\ldots W^{-1}W^{-*}W^*W\,W^*W\ldots W^*Z$$

$$= W^*W + Z^*Z$$

$$= \begin{bmatrix} W \\ Z \end{bmatrix}^* \begin{bmatrix} W \\ Z \end{bmatrix}$$

$$= I$$

When n is even, then

$A = W^*W\,W^*W\,W\ldots W^*W\ Z$, and set $X = W^*W$, we then need to show that X (which is positive definite) is a solution to the operator equation (1) .we have.

$$
\begin{aligned}
X + A^* X^{-n} A &= W^*W + \left(W^*W\, W^*W \ldots W^*W\, Z\right)^* \left(W^*W\right)^{-n} \left(W^*W\, W^*W \ldots W^*W\, Z\right) \\
&= W^*W + Z^*W^*W\, W^*W \ldots W^*W \left(W^*W\right)^{-1} \left(W^*W\right)^{-1} \ldots \left(W\ W\right)^{-1} \left(W^*W\, W^*W \ldots W^*W\, Z\right) \\
&= W^*W + Z^*W^*W\, W^*W \ldots W^*W\, W^{-1}W^{-*} \ldots W^{-1}W^{-*} \left(W^*W\, W^*W \ldots W^*WZ\right) \\
&= W^*W + Z^*Z \\
&= \begin{bmatrix} W \\ Z \end{bmatrix}^* \begin{bmatrix} W \\ Z \end{bmatrix} \\
&= I
\end{aligned}
$$

which completes the proof of the theorem.

3- Relation between solution X and operator A:

In this section, we will study the relations between X and A in equation (1)

Theorem 3.1: If equation (1) has a solution X, then for all $n \in N$ the following hold:

(i) $r\left(X^{\frac{-n}{2}+\frac{1}{2}} A - A^* X^{\frac{-n}{2}+\frac{1}{2}} \right) \le 1$.

(ii) $\left(X\right)^{\frac{n}{2}} \left(X^*\right)^{\frac{n}{2}} > A\ A^*$.

Proof:

(i) Using theorem (2.1), when n is even. We obtain:

$$
\begin{aligned}
r\left(X^{\frac{-n}{2}+\frac{1}{2}} A - A^* X^{\frac{-n}{2}+\frac{1}{2}} \right) &= r\left(\left(W^*W\right)^{\frac{-n}{2}+\frac{1}{2}} \left(W^*W\right)^{\frac{n}{2}} Z - Z^* \left(W^*W\right)^{\frac{n}{2}} \left(W^*W\right)^{\frac{-n}{2}+\frac{1}{2}} \right) \\
&= r\left(\left(W^*W\right)^{\frac{1}{2}} Z - Z^* \left(W^*W\right)^{\frac{1}{2}} \right)
\end{aligned}
$$

We set $M := \left(W^*W\right)^{\frac{1}{2}}$; then applying lemma (1.1), we obtain:

$$
\begin{aligned}
r\left(X^{\frac{-n}{2}+\frac{1}{2}} A - A^* X^{\frac{-n}{2}+\frac{1}{2}} \right) &= r\left(M^*Z - Z^*M\right) \\
&\le r\left(M^*M + Z^*Z\right) \\
&= r\left(I\right) \\
&= 1
\end{aligned}
$$

Now, when n is odd; we obtain

$$
\begin{aligned}
r\left(X^{\frac{-n}{2}+\frac{1}{2}} A - A^* X^{\frac{-n}{2}+\frac{1}{2}} \right) &= r\left(\left(W^*W\right)^{\frac{-n}{2}+\frac{1}{2}} \left(W^*W\right)^{\frac{n-1}{2}} W^*\ Z - Z^*\ W \left(W^*W\right)^{\frac{n-1}{2}} \left(W^*W\right)^{\frac{-n}{2}+\frac{1}{2}} \right) \\
&= r\left(W^*\ Z - Z^*\ W\right)
\end{aligned}
$$

then applying lemma (1.1) we obtain:

$$r\left(X^{\frac{-n}{2}+\frac{1}{2}} A - A^* X^{\frac{-n}{2}+\frac{1}{2}} \right) = r\left(W^* Z - Z^* W\right)$$

$$\leq r\left(W^* W + Z^* Z\right)$$

$$\leq r(I)$$

$$\leq 1$$

(ii) If n is even, then from theorem (2.1), we have

$$(X)^{\frac{n}{2}} (X^*)^{\frac{n}{2}} - A A^* = (W^* W)^{\frac{n}{2}} (W^* W)^{\frac{n}{2}} - (W^* W)^{\frac{n}{2}} Z Z^* (W^* W)^{\frac{n}{2}}$$

$$= (W^* W)^{\frac{n}{2}} (I - ZZ^*)(W^* W)^{\frac{n}{2}}$$

Since $W^* W + Z^* Z = I$, $spec(ZZ^*) = spec(Z^* Z)$ and, $I - Z^* Z > 0$, therefore, $(W^* W)^{\frac{n}{2}} (I - ZZ^*)(W^* W)^{\frac{n}{2}} > 0$.

If n is odd, then. From theorem (2.1), we have

$$(X)^{\frac{n}{2}} (X^*)^{\frac{n}{2}} - A A^* = (W^* W)^{\frac{n}{2}} (W^* W)^{\frac{n}{2}} - (W^* W)^{\frac{n-1}{2}} W^* Z Z^* W (W^* W)^{\frac{n-1}{2}}$$

$$= (W^* W)^{\frac{n-1}{2}} \left((W^* W)^{\frac{1}{2}} (W^* W)^{\frac{1}{2}} - W^* ZZ^* W \right) (W^* W)^{\frac{n-1}{2}}$$

$$= (W^* W)^{\frac{n-1}{2}} \left[W^* W - W^* ZZ^* W\right] (W^* W)^{\frac{n-1}{2}}$$

$$= (W^* W)^{\frac{n-1}{2}} W^* \left[I - ZZ^*\right] (W^* W)^{\frac{n-1}{2}} W$$

Since $W^* W + Z^* Z = I$ and $spec(ZZ^*) = spec(Z^* Z)$, $I - Z^* Z = W^* W > 0$, and thus,, $I - Z^* Z > 0$, therefore,, $(W^* W)^{\frac{n}{2}} (I - ZZ^*)(W^* W)^{\frac{n}{2}} > 0$

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