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# Finite Difference Method for Solving Fractional Hyperbolic Partial Differential Equations

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## Abstract

In this paper, the finite difference method is used to solve fractional hyperbolic partial differential equations, by modifying the associated explicit and implicit difference methods used to solve fractional partial differential equation. A comparison with the exact solution is presented and the results are given in tabulated form in order to give a good comparison with the exact solution.

## Introduction

An important type of differential equations which is called fractional differential equations in which the differintegration is of non-integer order [1].

Real life problems with fractional differential equations are of great importance, since fractional differential equations accumulate the whole information of the function in a weighted form. This has many applications in physics, chemistry, engineering ,etc. For that reason, we need a method for solving such equations, effectively, easy use and applied for different problems[2].

Consider the fractional order partial differential equation [3][4]:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c(x,t)\frac{\partial^q u(x,t)}{\partial x^q} + s(x,t),\ L \le x \le R,\ 0 \le t \le T \qquad \text{.............} (1)$$

together with the initial and zero Dirichlet boundary conditions :

$$\left.\begin{array}{l} u(x,0) = f(x), u_t(x,0)=h(x)\ ,\ L \le x \le R \\ u(L,t) = 0,\ u(R,t) = 0 \text{ for } 0 \le t \le T \end{array}\right\} \qquad \text{..........} (2)$$

where $\frac{\partial^q u(x,t)}{\partial x^q}$ denote the left-hand partial fractional derivative of order q of the function u with

respect to x and $1 < q \le 2$ .

The left-handed shifted and the right-handed shifted Grünwald estimate to the left-handed and right-handed derivatives, are given by [1][5][6] :

$$\frac{d^q f(x)}{d_+ x^q} = \frac{1}{(\Delta x)^q}\sum_{k=0}^{n} g_k f(x-(k-1)\Delta x)$$

$$\frac{d^q f(x)}{d_- x^q} = \frac{1}{(\Delta x)^q} \sum_{k=0}^{n} g_k f(x + (k-1)\,\Delta x)$$

where n is the number of subdivisions of the interval [ L, R ] and q is a fractional number. Therefore:

$$\frac{\partial^q u(x_i, t_j)}{\partial_+ x^q} = \frac{1}{(\Delta x)^q} \sum_{k=0}^{i+1} g_k u(x_i - (k-1)\Delta x, t_j)$$

$$= \frac{1}{(\Delta x)^q} \sum_{k=0}^{i+1} g_k u_{i-k+1,} \qquad \text{……………..(3)}$$

and

$$\frac{\partial^q u(x_i, t_j)}{\partial_- x^q} = \frac{1}{(\Delta x)^q} \sum_{k=0}^{n-i+1} g_k u(x_i + (k-1)\,\Delta x, t_j)$$

$$= \frac{1}{(\Delta x)^q} \sum_{k=0}^{n-i+1} g_k u_{i+k-1,j} \qquad \text{........................ (4)}$$

where $g_0 = 1$ and $g_k = (-1)^k \dfrac{q(q-1)\ldots(q-k+1)}{k!}$, $k = 1,2,\ldots$

## The Explicit Finite Difference Method for Solving Fractional Hyperbolic Partial Differential Equations

The explicit finite difference method is improved to solve the initial-boundary value problem (1)-(2). To do this, we substitute $t = t_j$, in eq. (1) and replace the partial derivative $\dfrac{\partial^2 u}{\partial t^2}$ with its central difference approximation to get :

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta t)^2} = c_{i,j} \frac{\partial^q u_{i,j}}{\partial x^q} + s_{i,j} \qquad \text{...................... (5)}$$

where $t_j = j\Delta t$, $j=0,1,\ldots,m$ and m is the number of subdivisions of the interval [0, T], $t \in \mathrm{R}$.

Next, substitute equation (3) in equation (5) to obtain:

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta t)^2} = \frac{c_{i,j}}{(\Delta x)^q} \sum_{k=0}^{i+1} g_k u_{i-k+1,j} + s_{i,j}, i = 1,2,\ldots,n-1; j = 0,1,\ldots,m-1 \quad \ldots(6)$$

On the other hand, the initial and boundary conditions given by eq.(2) becomes :

$$u_{i,0} = u(x_i, 0) = f(x_i),\ \frac{\partial u(x_i, 0)}{\partial t} \text{ for } i = 0,1,\ldots,n$$

$$u_{0,j} = u(L, t_j) = 0,\ u_{n,j} = u(R,t) = 0 \text{ for } j = 0,1,\ldots,m$$

and by using the central difference approximation to the initial derivative conditions ,one can get :

$$\frac{1}{2\Delta t}(u_{i,1}-u_{i,-1})=h_i\text{ , i=0,1,…,n}$$

where $h_i = h(x_i)$ for i=0, 1,…, n. Hence :

$$u_{i,1}=u_{i,-1}+2\Delta t h_i\text{ , i=0,1,…, n}$$

Moreover, equation (6) becomes:

$$u_{i,j+1}=2u_{i,j}-u_{i,j-1}+\frac{(\Delta t)^2 c_{i,j}}{(\Delta x)^q}\sum_{k=0}^{i+1} g_k u_{i-k+1,j}+s_{i,j}(\Delta t)^2 \quad \text{……………….(7)}$$

where i=1,2,…,n−1, j=0,1,…,m−1.

Therefore:

$$u_{i,1}=2u_{i,0}-u_{i,-1}+\frac{(\Delta t)^2 c_{i,0}}{(\Delta x)^q}\sum_{k=0}^{i+1} g_k u_{i-k+1,0}+s_{i,0}(\Delta t)^2 \quad \text{……..……….(8)}$$

By substituting $u_{i,-1}=u_{i,1}-2\Delta t h_i$ back into eq.(8) one can show that $u_{i,1}$ can be calculated from the following equation:

$$u_{i,1}=f_i+\frac{(\Delta t)^2 c_{i,0}}{2(\Delta x)^q}\sum_{k=0}^{i+1} g_k f_{i-k+1}+\frac{(\Delta t)^2}{2}s_{i,0}+\Delta t g_i\text{ , i=0,1,……..n −1}$$

where i=0, 1,….. n−1.

By evaluating the above equation for each i=0,1,…,n−1, one can get the values of $u_{i,1}$, $i=1,2,...,n-1$. Then by evaluating equation (7) at each i=1,2,…,n−1 and j=2,3,…,m−1 one can get the numerical solution of eq.(1).

Then the resulting equation can be explicitly solved to give:

$$u_{i,j+1}-2u_{i,j}+u_{i,j-1}=r\sum_{w=0}^{i+1} g_w u_{i-w+1,j} \quad \text{………….……..(9)}$$

Where $r = k^2/h^q$ .The resulting difference equation is stable since we

let $g_0$=1 and $g_w = (-1)\dfrac{wq(q-1)k(q-w-1)}{w}$ , w = 1,2,…..

$1 \le q \le 2$ , i≠1 ,hence $g_i \ge 0$ for all value of i. Therefore:

$$\sum_{w=0}^{i+1} g_w \le -g_1 = -(-q) = q \quad \text{………………(10)}$$

The difference between the analytical and numerical solutions of the difference equation remains bounded as j increases.

Let the error $E_{i,j}$=u($h_i$, $k_j$) $-u_{i,j}$ then the stability condition under which the finite difference eq. (9)

is stable, to find the stability conditions under which the error $E_{i,j}$ is bounded .

Smith [7] shows that the error $E_{i,j}$ can be written in the form :

$E_{i,j}$= $e^{\gamma\beta ih}\xi^j$, where s= $e^{\alpha k}$ $\gamma = \sqrt{-1}$ ......................(11)

Where α is a complex constant, one can substitute eqs .(10), (11) into (9), to get:

$$\varepsilon - 2 - \varepsilon^{-1} - rqe^{\gamma\beta h(1-w)} \leq 0$$

Assuming that, $\theta = \beta h(1-w)$,then it is easily known that the equation for R is:

$$\varepsilon^2 - (2 - rqe^{\gamma\theta})\varepsilon + 1 = 0$$

Let $A = 2 + rqe^{\gamma\theta}$, where $\left|e^{\gamma\theta}\right| \leq 1$

Hence the values of $\varepsilon$ are:

$$\varepsilon_1 = \frac{A + \sqrt{A^2 - 4}}{2} \text{ and } \varepsilon_2 = \frac{A - \sqrt{A^2 + 4}}{2}$$

From eq.(11), the error will not grow with time if

$\left|\varepsilon^\gamma\right| \leq 1$, for all real β .................. (12)

Equation (12) is called the Von-Neumann's condition for stability .Thus we will use eq. (12) to

find the stability condition of the finite difference equation.

For stability; as r, q and β are real and when giving stability while $\varepsilon_2$ gives instability. When $-1 \leq A \leq 1$ , we get $\varepsilon_1 \text{and} \varepsilon_2$ are complex number, hence:

$$\varepsilon_1 = \frac{A + y\sqrt{4 - A^2}}{2} \text{ and } \varepsilon_1 = \frac{A - y\sqrt{4 - A^2}}{2}$$

Then using Von-Neumann's condition (12) to prove that eq.(9) is stable

For $-1 \leq A \leq 1$, the only useful inequality is $A \leq -1$, hence $2 + rqe^{\gamma\theta} \leq 1$, where $\left|e^{\gamma\theta}\right| \leq 1$.Therefore; $r \leq \frac{-1}{q}$ , where $1 \leq q \leq 2$.

Hence, $|r| \leq \frac{1}{2}$ , which is the stability condition.

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## The Implicit Finite Difference Method for Solving Fractional Hyperbolic Partial Differential Equations

Now, we can improve and introduce similar approach for the implicit finite difference method

to solved the one–sided fractional hyperbolic partial differential equations. The resulting discretization takes the following form:

$$\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2}=\frac{c_{i,j}}{h^q}\sum_{w=0}^{i+1}g_w u_{i-w+1,j}$$

Where $i=1,2,...,n-1; j=0,1,...,m-1$. Then to get

$$u_{i,j+1}-2u_{i,j}+u_{i,j-1}=r\sum_{w=0}^{i+1}g_w u_{i-w+1,j+1} \qquad \text{.......................(13)}$$

In the above equation and under the same conditions of eq.(9) and substituting eqs. (10) and (11) into eq. (13), one can get:

$\varepsilon-2-\varepsilon^{-1}< rqe^{\gamma\theta}\varepsilon$, where $\theta=\beta h(1-w)$.

Hence the values of $\varepsilon$ are :

$$\varepsilon_1=\frac{1+(1-A)^{\frac{1}{2}}}{A} \text{ and } \varepsilon_2=\frac{1-(1-A)^{\frac{1}{2}}}{A} \text{ where } A=1-rqe^{\gamma\theta}.$$

To discuss the stability of eq. (13); by using Von-Neumann's condition (12). When A $<-1$,

we get real the roots, $\varepsilon_1$ also, which gives instability while $\varepsilon_2$ gives stability for this problem .

Now, when $-1\le A\le 1$, we get complex number, which are $\varepsilon_1=\dfrac{1-\gamma(1-A)^{\frac{1}{2}}}{A}$

and $\varepsilon_1=\dfrac{1+\gamma(1-A)^{\frac{1}{2}}}{A}$ .and the condition of the stability leads to $r\ge 1$ when $1\le q\le 2$ and

$$\left|e^{\gamma\theta}\right|\le 1$$

Therefore; the finite difference eq. (13) is instable for $r \le \dfrac{2}{q}$, $1\le q\le 2$.

## Illustrative Example

To illustrate the methods of the solution, an illustrative numerical example is considered:

Example:-



Consider the fractional order partial differential equation :

$$\frac{\partial^2 u}{\partial t^2}=\frac{1}{\Gamma(0.5)}x^{\frac{1}{2}}\frac{\partial^{1.5}u}{\partial x^{1.5}}-4x^2+2x^3-2.546x^2t^2+2.546xt^2,\ 0\le x\le 2,\ 0\le t\le 1$$

Together with initial and zero Dirichlet boundary conditions:

u(x, 0)=0, $\frac{\partial u(x,0)}{\partial t}=0$ for $0\le x\le 2$..

u(0, t)=0, u(1, t)=0 for $0\le t\le 1$.

This example has the exact solution as: $u(x,t)=x^2(x-2)t^2$,[8]. which is considered for the comparison purpose. Here; we use the explicit and implicit finite difference methods to solve this example numerically. To do this, first we divide the x-interval into 2 subintervals such that $x_i=i$, i=0,1,2 and the t-interval into 2 subintervals such that $t_j=\frac{j}{2}$, j=0,1,2. Thus, the initial and zero Dirichlet boundary conditions become:

$u(x_i,0)=0$ for i=0,1,2.

$\frac{\partial u(x_i,0)}{\partial t}=0$ for i=0,1,2.

$u(0,t_j)=0$ for j=0,1,2.

$u(1,t_j)=0$ for j=0,1,2.

By using the central difference approximation to the initial derivative condition one can get:

$\frac{1}{2\Delta t}(u_{i,1}-u_{i,-1})=0$; hence

$u_{i,1}=u_{i,-1}$ for i=0,1,2.

Moreover, equation (7) becomes:

$$u_{i,j+1}=2u_{i,j}-u_{i,j-1}+0.25x_i^{\frac{1}{2}}\sum_{k=0}^{i+1}g_k u_{i-k+1,j}$$
$$+0.25(-4x_i^2+2x_i^3-2.546x_i^2t_j^2+2.546x_it_j^2)$$

where i=1 and j=0, 1.

Therefore

$$u_{i,1}=2u_{i,0}-u_{i,-1}+0.25x_i^{\frac{1}{2}}\sum_{k=0}^{i+1}g_k u_{i-k+1,0}+$$
$$0.25(-4x_i^2+2x_i^3-2.546x_i^2t_0^2+2.546xt_0^2)$$

By substituting $u_{i,-1} = u_{i,1}$ in the above equation one can show that $u_{i,1}$ can be calculated from the equation

$$u_{i,1} = u_{i,0} + 0.125 x_i^{\frac{1}{2}} \sum_{k=0}^{i+1} g_k u_{i-k+1,0} +$$

$$0.125(-4x_i^2 + 2x_i^3 - 2.546 x_i^2 t_0^2 + 2.546 x t_0^2)$$

Thus

$$u_{1,1} = 0.125(-4x_1^2 + 2x_1^3) = -0.25.$$

Then

$$u_{1,2} = 2u_{1,1} - u_{1,0} + 0.25 x_1^{\frac{1}{2}} \sum_{k=0}^{2} g_k u_{2-k,1} +$$

$$0.25(-4x_1^2 + 2x_1^3 - 2.546 x_1^2 t_1^2 + 2.546 x_1 t_1^2) = -0.947.$$

These values are tabulated down with the comparison with the exact solution. See table (1)

Second, we divide the x-interval into 10 and the t-interval into 10 subinterval. Thus, the initial and zero Dirichlet boundary conditions become:

$$u(x_i, 0) = 0,\ \frac{\partial u(x_i, 0)}{\partial t} = 0 \text{ for } i = 0, 1, \ldots, 10.$$

$$u(0, t_j) = 0,\ u(1, t_j) = 0, \text{ for } j = 0, 1, \ldots, 10.$$

The results are presented in table (2).

## Conclusions

**1.** The finite difference method gave the numerical solution of the fractional differential equations and it depended on the Grunwald estimate for the fractional derivatives .
**2.** The stability results in the finite partial differential equation case as generalization and unification for the corresponding result in the classical hyperbolic partial differential equation.
**3.** Similar to this work, the explicit finite difference method can be also used to solve the initial-boundary value problems of the two-sided fractional hyperbolic partial differential equations given by, $\frac{\partial u(x,t)}{\partial t} = c(x,t)\frac{\partial^q u(x,t)}{\partial_+ x^q} + d(x,t)\frac{\partial^q u(x,t)}{\partial_- x^q} + s(x,t)$

together with the initial and zero Dirichlet boundary conditions:

$$u(x,0) = f(x), u(L,t) = 0, u(R,t) = 0 \quad \text{for } L \le x \le R\}$$

where $L \le x \le R$, $0 \le t \le T$, $\frac{\partial^q u(x,t)}{\partial_+ x^q}$ and $\frac{\partial^q u(x,t)}{\partial_- x^q}$ denote the left-handed and the right- handed partial fractional derivatives of order q of the function u with respect to x and $1 < q \le 2$.

In this case equation (4) becomes

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \frac{c_{i,j+1}}{(\Delta x)^q} \sum_{w=0}^{i+1} g_w u_{i-w+1,j} + \frac{d_{i,j+1}}{(\Delta x)^q} \sum_{w=0}^{n-i+1} g_w u_{i+w-1,j} + s_{i,j}$$

**4.** In a similar manner, the implicit finite difference method can be also used to solve the initial-boundary value problems of the two-sided fractional hyperbolic partial differential equations given by equations:-

$$\frac{\partial u(x,t)}{\partial t} = c(x,t)\frac{\partial^q u(x,t)}{\partial_+ x^q} + d(x,t)\frac{\partial^q u(x,t)}{\partial_- x^q} + s(x,t)$$

In this case eq.(4)becomes:

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \frac{c_{i,j+1}}{(\Delta x)^q} \sum_{w=0}^{i+1} g_w u_{i-w+1,j+1} + \frac{d_{i,j+1}}{(\Delta x)^q} \sum_{w=0}^{n-i+1} g_w u_{i+w-1,j+1} + s_{i,j+1}$$

where i=1,2,…, n−1; j=0,1,…,m−1.

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**Table (1) Represents the numerical and the exact solutions for n=m=2 of example.**

| $x_i$ | $t_j$ | Numerical solution $u_{i,j}$ | | Exact solution $u(x_i,t_j)$ |
|---|---|---|---|---|
| | | Explicit method | Implicit method | |
| 1 | 0.5 | −0.25 | −0.25 | −0.25 |
| 1 | 1 | −0.9472 | −0.9684 | −1 |

**Table (2) Represents the numerical and the exact solutions for n=m=10 of example.**

| $x_i$ | $t_j$ | Numerical solution $u_{i,j}$ | | Exact solution $u(x_i,t_j)$ |
|---|---|---|---|---|
| | | Explicit method | Implicit method | |
| 1 | 0.5 | −0.25 | −0.25 | −0.25 |
| 1 | 1 | −0.994 | −0.995 | −1 |
| 0.8 | 0.2 | −0.0398 | −0.0389 | −0.031 |
| 0.2 | 0.7 | −0.0326 | −0.0393 | −0.035 |
| 0.4 | 0.9 | −0.2129 | −0.2026 | −0.207 |
| 0.6 | 1 | −0. 5096 | −0. 5063 | −0. 504 |
| 1.2 | 0.7 | −0.5655 | −0.5641 | −0.564 |
| 1.4 | 0.3 | −0.1014 | −0.1042 | −0.106 |
| 1.6 | 0.8 | −0.6568 | −0.6551 | −0.655 |
| 1.8 | 1 | −0.6466 | −0.6467 | −0.648 |

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# طرائق الفروقات المنتهية لحل المعادلات التفاضلية الجزئية الكسرية من نوع القطع الزائد

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## الخلاصة

في هذا البحث ، استخدمت طريقة الفروقات المنتهية ( finite difference method) لحل المعادلات التفاضلية الجزئية ذي الرتب الكسرية من نوع القطع الزائد ( Hyperbolic partial differational equation) ،بتطوير طريقة الفروقات المنتهية الصريحة والضمنية ( Explicit and Implicit method) .

قورنت النتائج العددية مع الحل الصحيح وأعطيت النتائج في جداول للحصول على أفضل مقارنة مع الحل الصحيح .