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On Fuzzy Internal Direct Product

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Abstract

The main aim of this paper is to introduce the concept of a Fuzzy Internal Direct Product of fuzzy subgroups of group . We study some properties and prove some theorems about this concept ,which is very important and interesting of fuzzy groups and very useful in applications of fuzzy mathematics in general and especially in fuzzy groups.

Introduction

Applying the concept of fuzzy sets of Zadeh to the group theory, Rosenfeld introduced the notion of a fuzzy group as early as 1971.

The technique of generating a fuzzy group (the smallest fuzzy group) containing an arbitrarily chosen fuzzy set was developed only in 1992 by Malik , Mordeson and Nair, [1].

In this paper, we use our notion of fuzzy inner product to generate Fuzzy Internal Direct Product of fuzzy subgroups of group.

Now we introduce the following definitions which is necessary and needed in the next section :

Definition 1.1 [1], [2]:

A mapping from a nonempty set X to the interval [0, 1] is called a fuzzy subset of X .

Next, we shall give some definitions and concepts related to fuzzy subsets of G.

Definition 1.2:

Let μ, v be fuzzy subsets of G, if $\mu(x) \leq v(x)$ for every $x \in G$, then we say that μ is contained in v (or v contains μ) and we write $\mu \subseteq v$ (or $v \supseteq \mu$).

If $\mu \subseteq v$ and $\mu \neq v$, then μ is said to be properly contained in v (or v properly contains μ) and we write $\mu \subset v$ (or $v \supset \mu$).[3]

Note that: $\mu = v$ if and only if $\mu(x) = v(x)$ for all $x \in G$.[4]

Definition 1.3 [3] :

Let μ, v be two fuzzy subsets of G. Then $\mu \cup v$ *and* $\mu \cap v$ are fuzzy subsets as follows:

(i) $(\mu \cup v)(x) = \max\{\mu(x), v(x)\}$

(ii) $(\mu \cap v)(x) = \min\{\mu(x), v(x)\}$, for all $x \in G$

Then $\mu \cup v$ *and* $\mu \cap v$ are called the union and intersection of μ and v, respectively.

Now, we are ready to give the definition of a fuzzy subgroup of a group.

Definition 1.4[1], [5]**:**

A fuzzy subset μ of a group G is a fuzzy subgroup of G if:

(i) $min\{\mu(a), \mu(b)\} \leq \mu(a * b)$

(ii) $\mu(a^{-1}) = \mu(a)$, for all $a, b \in G$.

Proposition 1.5 [6]:

Let μ be a fuzzy group. Then $\mu(a) \leq \mu(e) \quad \forall a \in G$.

Definition 1.6 [7]:

If μ is a fuzzy subgroup of G, then μ is said to be abelian if $\forall\ x, y \in G$, $\mu(x) > 0, \mu(y) > 0$, then $\mu(xy) = \mu(yx)$.

Definition 1.7 [8] , [9]:

A fuzzy subgroup μ of G is said to be normal fuzzy subgroup if $\mu(x * y) = \mu(y * x)$, $\forall x, y \in G$.

Fuzzy Internal Direct Product

Now we are ready to introduce the definition and some theorems about fuzzy internal direct product of fuzzy sub groups of groups :

Definition 2.1

Let A be a fuzzy subgroup of a group (G,.) and $N_1,N_2,\ldots,N_n$ be fuzzy normal subgroups in A such that :

1- $A = N_1 \times N_2 \times \ldots \times N_n$

2- Let $x_t \subseteq A$, $x \in G$, $t \in [0, 1]$, then :

$$x_t(y) = \begin{cases} \sup\{\min\{n_1(y_1), n_2(y_2), \ldots, n_n(y_n)\} \\ \qquad y = y_1 \cdot y_2 \ldots y_n \text{ and } y_1, y_2, \ldots, y_n \in G \quad \text{if } y = x \\ 0 \qquad \text{otherwise} \end{cases}$$

$n_i \subseteq N_i$ in unique way.

Then A is said to be fuzzy internal direct product of $N_1,N_2,\ldots,N_n$.

Now we introduce the following theorem :

Theorem 2.2

If a fuzzy subgroup A of a group (G, .) is the fuzzy internal direct product of fuzzy normal subgroups : $N_1,N_2,\ldots,N_n$, then for $i \neq j$, $N_i \cap N_j = \{e_t\}$, and if $n_i \subseteq N_i$, $n_j \subseteq N_j$ then $n_i n_j = n_j n_i$.

Proof:

Suppose that $n_s \subseteq N_i \cap N_j$ then we can write n_s as

$$n_s(y) = \begin{cases} \sup\{\min\ \{e_1, \ldots, e_{i-1}, n_s(y), e_{i+1} \ldots, e_j, \ldots, e_n\}\} \text{ if } y = n, n_i \subseteq N_i \ \ i = 1, \ldots, n \\ 0 \qquad \text{other wise} \end{cases}$$

Viewing n_s a an fuzzy singleton in N_i . similarly, we can write n_s as

$$n_s(y) = \begin{cases} \sup\{\min\ \{e_1, \ldots, e_i, \ldots, e_{j-1}, n_s(y), e_{j+1}, \ldots, e_n\}\} \text{ if } y = n, n_i \subseteq N_i \ \ i = 1, ..n \\ 0 \qquad \text{other wise} \end{cases}$$

Since the two decompositions in this form for n_s must coincide, the entry from N_i in each must be equal. In the first decomposition this entry is n_s, in the other it is e_t; hence $n_s = e_t$. Thus $N_i \cap N_j = \{e_t\}$ for $i \neq j$.

Suppose $n_i \subseteq N_i$, $n_j \subseteq N_j$, and $i \neq j$ then $n_i n_j n_i^{-1} \in N_j$ since N_j is normal fuzzy; thus $n_i n_j n_i^{-1} n_j^{-1} \subseteq N_j$. Similarly, since $n_i^{-1} \subseteq N_i$, $n_j n_i^{-1} n_j^{-1} \subseteq N_i$, whence $n_i n_j n_i^{-1} n_j^{-1} \subseteq N_i$ then $n_i n_j n_i^{-1} n_j^{-1} \subseteq N_i \cap N_j = \{e_t\}$.

Thus $n_i n_j n_i^{-1} n_j^{-1} = e_t$; this gives the desired result $n_i n_j = n_j n_i$.

Remark

One should point out that if $k_1,\ldots,k_n$ are normal fuzzy subgroups of A, such that $A = k_1 \times k_2 \times \ldots \times k_n$ and $k_i \cap k_j = \{e_t\}$ for $i \neq j$ it need not be true that A is the fuzzy internal direct product of $k_1,\ldots,k_n$. A more stringent condition is needed.

Clearly from the above theorem we can obtain the following corollary :

Corollary 2.3

A fuzzy subgroup A of a group (G, .) is the fuzzy internal direct product of the normal fuzzy subgroups $N_1,\ldots,N_n$ if and only if :

(1) $A = N_1 \times N_2 \times \ldots \times N_n$

(2) $N_i \cap (N_1 \times N_2 \times \ldots \times N_{i-1} \times N_{i+1} \times \ldots \times N_n) = \{e\}$ for $i = 1,\ldots,n$

Definition 2.4

Let H and K are fuzzy subgroups of a group (G, .) .The join $H \vee k$ of H and k is the intersection of all fuzzy subgroups of G containing H.K .

$$H.K = \begin{cases} \sup\{\min\ \{H(x_1), K(x_2)\} x = x_1 \cdot x_2, x_1, x_2 \in G\} & if\ \ x \in \mathrm{Im}(\cdot) \\ 0 & other\ wise \end{cases}$$

Clearly, this intersection will be the smallest possible fuzzy subgroup of G containing H.K , and if elements in H and K commute, in particular, if G is abelian, we have $H \vee K = H.K$,since $H . K \subseteq H \vee K$ (by definition 2.4) and since :

$H(x_1) = \sup\{\min \{H(x_1), K(e)\}\ x_1 \in G\}$ and

$K(x_2) = \sup\ \{\min\ \{H(e), K(x_2)\}\ x_2 \in G\}$, then :

$H \subseteq H . K$ and $K \subseteq H . K$, so $H \vee K \subseteq H . K$, thus $H \vee K = H.K$.

Note that

Clearly, $H \vee K$ would be contained in any fuzzy subgroup containing both H and K .

Thus we see that $H \vee K$ is the smallest fuzzy subgroup of G containing both H and K .

Theorem 2.5

A fuzzy subgroup A of a group (G, .) is the fuzzy internal direct product of fuzzy subgroups H and K if and only if :

1) $A = H \vee K$

2) $x_t \cdot y_s = y_s \cdot x_t$ for all $x_t \subseteq H$ and $y_s \subseteq K$, t, s $\in [0,1]$, x, y $\in G$.

3) $H \cap K = \{e\}$

Proof:

Let A be the fuzzy internal direct product of H and K. we claim that (1), (2) and (3) are obvious if one will regard A as isomorphic to the fuzzy internal direct product of H and K under the map ϕ, with $\phi(x_t, y_s) = x_t \cdot y_s = (x \cdot y)_r$ where r = min { t, s} , x, y $\in$ G, r, s $\in$ [0, 1].

Under this map

$\overline{H} = \{(x_t, e) \mid x_t \subseteq H\}$ Corresponds to H , and

$\overline{K} = \{(e, y_s) \mid y_s \subseteq K\}$ Corresponds to K .

Then (1), (2) and (3) follow immediately from the corresponding assertions regarding $\overline{H}$ *and* $\overline{K}$ in $H \times K$, which are obvious.

Conversely, let (1), (2) and (3) hold. We must show that the map ϕ of the fuzzy internal direct product $H \times K$ in to A, given by :

$\phi(x_t, y_s) = x_t \cdot y_s = (x.y)_r$ where r = min{t, s}, x, y $\in$ G, r, s $\in$ [0,1], is an isomorphism. The map ϕ has already been defined suppose $\phi(x_{t1}, y_{s1}) = \phi(x_{t2}, y_{s2})$.

Then $x_{t1} \cdot y_{s1} = x_{t2} \cdot y_{s2}$; consequently $x_{t2}^{-1} \cdot x_{t1} = y_{s2} \cdot y_{s1}^{-1}$

But $x^{-1}_{t2} \cdot x_{t1} \subseteq H$ and $y_{s2} \cdot y^{-1}_{s1} \subseteq K$ and they are the same element and thus in $H \cap K = \{e\}$ by (3). Therefore, $x^{-1}_{t2} \cdot x_{t1} = e$ and $x_{t1} = x_{t2}$.

Likewise, $y_{s1} = y_{s2}$, so $(x_{t1}, y_{s1}) = (x_{t2}, y_{s2})$. This shows that ϕ is one to one. The fact that $x_t \cdot y_s = y_s \cdot x_t$ by (2) for all $x_t \subseteq H$ and $y_s \subseteq k$ means that

$H.k(x) = \{\sup\{ \min\{H(x_1), k(x_2)\}\}; x = x_1 . x_2 , x_1, x_2 \in G\}$ is a fuzzy subgroup, for we have seen that this is the case if fuzzy singletons of H commute with those of K .

Thus by (1), $H. K = H \vee K = A$, so ϕ is on to A . Finally,

$\phi[(x_{t1}, y_{s1})(x_{t2}, y_{s2})] = \phi(x_{t1}.x_{t2}, y_{s1}.y_{s2}) = x_{t1}.x_{t2}.y_{s1}.y_{s2}$, while $[\phi(x_{t1},y_{s1})][\phi(x_{t2},y_{s2})] = x_{t1}.y_{s1}.x_{t2}.y_{s2}$.

But by (2) we have $y_{s1} \cdot x_{t2} = x_{t2} \cdot y_{s1}$. Thus :

$\phi[(x_{t1},y_{s1})(x_{t2},y_{s2})] = [\phi(x_{t1},y_{s1})][\phi(x_{t2}, y_{s2})]$.

Remark:

Not every fuzzy subgroup of abelian group is the fuzzy internal direct product of two proper fuzzy subgroups.

The following corollary is immediate consequences of the above theorem.

Corollary 2.6

Let A be fuzzy subgroup of a group (G, .). Let x_t and y_s be fuzzy singletons of A which commute and are of relatively prime orders r and s and $<x_t>$, $<y_s>$ are fuzzy subgroups of $<x_t> \vee <y_s>$. Then $x_t.y_s$ is of order r.s .

Fuzzy Invariants Of Fuzzy Subgroup

In this section we introduce the following definition and Lemma about fuzzy invariants of fuzzy subgroup :

Definition 3.1

Let A be fuzzy subgroup of abelian group (G, .) of order p^n, p a prime, and $A = A_1 \times A_2 \times \ldots . \times A_k$ where each A_i is fuzzy generating set of order p^{ni} with $n_1 \geq n_2 \geq \ldots \geq n_k > 0$, then the integers $n_1, n_2, \ldots, n_k$ are called the fuzzy invariants of A just because we called the integers above the fuzzy invariants of A does not mean that they are really the fuzzy invariants of A. That is, it is possible to assign different sets of fuzzy invariants to A .

We shall soon show that the fuzzy invariants of A are indeed unique and completely describe A. Note one other thing about the fuzzy invariants of A. If $A = A_1 \times \ldots \times A_k$, where A_i is fuzzy generating set of order P^{ni}, $n_1 \geq n_2 \geq \ldots \geq n_k > 0$, then $o(A) = o(A_1)o(A_2) \ldots o(A_k)$, hence $P^n = P^{n1}P^{n2} \ldots P^{nk} = P^{n1+n2+\ldots+nk}$, whence $n = n_1+n_2+ \ldots + n_k$

In other words, $n_1, n_2, \ldots, n_k$ give us a partition of n .

Before discussing the uniqueness of the fuzzy invariants of A, one thing should be made absolutely clear the singleton fuzzy $a_1, \ldots, a_k$ and the fuzzy subgroups $A_1, \ldots, A_k$ which they generate, which a rose above to give the decomposition of A in to a fuzzy internal direct product of fuzzy generating subgroups, are not unique. Let's see this in a very simple example Let G = {e,a,b,a.b} be an abelian group of order 4 where

$a^2 = b^2 = e$, ab = ba and A(e) = A(a) = 1, A(b) = A(a.b) = 3/4.

Then $A = H \times K$ where $H = <a_t>$, $K = <b_t>$ are fuzzy generating subgroups of order 2. But we have another decomposition of A as a fuzzy internal direct product, namely $A = N \times K$ where $N = <ab>$ and $K = <b>$. So, even in this fuzzy subgroup of very small order, we can get distinct decompositions of the fuzzy subgroup as the internal direct product of fuzzy generating subgroups.

Lemma 3.2

Let A be fuzzy subgroup of abelian group (G, .) of order p^n, p a prime. Suppose that $A = A_1 \times A_2 \times \ldots . \times A_k$, where each $A_i = <a_{si}>$ is fuzzy generating of order p^{ni}, and $n_1 \geq n_2 \geq \ldots \geq n_k > 0$. If m is an integer such that $n_t > m \geq n_{t+1}$ then :

$A(p^m) = B_1 \times \ldots \times B_t \times A_{t+1} \times \ldots \times A_k$ where B_i is fuzzy generating of order p^m, generated by a_{si}, for $i \leq t$. The order of $A(p^m)$ is p^u, where

$$u = mt + \sum_{i=t+1}^{k} n_i$$

Proof:

First of all, we claim that $A_{t+1}, \ldots, A_k$ are all in $A(p^m)$, since $m \geq n_{t+1} \geq \ldots \geq n_k > 0$, if $j \geq t+1$,

$$a_j^{p^m} = (a_j^{p^{nj}})^{p^{m-nj}} = e$$

Hence A_j , for $j \geq t+1$ lies in $A(p^m)$.

Secondly, if $i \leq t$ then $n_i > m$ and $(a_i^{p^{ni-m}}) = a_i^{p^{ni}} = e$

whence each such $a_i^{p^{ni-m}}$ is in $A(p^m)$ and so the fuzzy subgroup it generates, B_i, is also in $A(p^m)$. Since $B_1, \ldots, B_t, A_{t+1}, \ldots, A_k$ all in $A(p^m)$, their product (which is fuzzy direct, since the product $A_1 \times A_2 \times \ldots \times A_k$ is fuzzy direct) is in $A(p^m)$.

Hence $A(p^m) \supset B_1 \times \ldots \times B_t \times A_{t+1} \times \ldots \times A_k$.

On the other hand, if :

$y_r = a_1^{\lambda 1}.a_2^{\lambda 2} \ldots a_k^{\lambda k}$ is in $A(p^m)$, since it then satisfies $y_r^{p^m} = e$, we set :

$$e = y_r^{p^m} = a_1^{\lambda 1 p^m} \ldots a_k^{\lambda k p^m}$$

However, the product of the fuzzy subgroups $A_1, \ldots, A_k$ is fuzzy direct, so we get :

$$a_1^{\lambda 1 p^m} = e, \ldots, a_k^{\lambda k p^m} = e$$

Thus the order of a_i, that is, p^{ni} must divide $\lambda_i p^m$ for $i = 1, 2, \ldots, k$.

If $i \geq t+1$ this is automatically true whatever be the choice of $\lambda_{t+1}, \ldots, \lambda_k$ since $m \geq n_{t+1} \geq \ldots \geq n_k$,

Hence $p^{ni} \mid p^m$, $i \geq t+1$. However, for $i \leq t$, we get from $p^{ni} \mid \lambda_i p^m$ that $p^{ni-m} \mid \lambda_i$, therefore $\lambda_i = v_i P^{ni-m}$ for some integer v_i .

Putting all this in formation in to the values of the λ_i's in the expression for y_r as :

$y_r = a_1^{\lambda 1} \ldots a_k^{\lambda k}$ We see that $y_r = a_1^{v1 p^{n1-m}} \ldots a_t^{vt p^{nt-m}} a_{t+1}^{\lambda t+1} \ldots a_k^{\lambda k}$

This says that $y_r \in B_1 \times \ldots \times B_t \times A_{t+1} \times \ldots \times A_k$.

Now since each B_i is of order p^m and since $o(A_i) = p^{ni}$ and since $A = B_1 \times \ldots \times B_t \times A_{t+1} \times \ldots \times A_k$,

$o(A) = o(B_1)\, o(B_2) \ldots o(B_t)\, o(A_{t+1}) \ldots o(A_k) = p^m p^m \ldots p^m\, p^{nt+1} \ldots p^{nk}$

Thus, if we write $o(A) = p^u$, then:

$u = mt + \sum_{i=t+1}^{k} n_i$ The lemma is proved.

Corollary 3.3

If A is a fuzzy subgroup of a group in lemma (3.2), then $o(A(p)) = p^k$.

Proof:

Apply the above lemma to the case $m = 1$.

Then $t = k$, hence $u = 1. k = k$ and so $o(A) = p^k$.

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حول الجداء المباشر الداخلي الضبابي

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الخلاصة

يتضمن البحث تقديم تعريف الجداء المباشر الداخلي الضبابي وبعض الخواص والمبرهنات المتعلقة به الذي يعد موضوعاً مهماً و ضرورياً وذا تطبيقات واسعة في مجال الرياضيات الضبابية بشكل عام وفي مجال الزمرة الضبابية بشكل خاص .